On similarity of quasinilpotent operators

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Abstract

Bounded linear operators on separable Banach spaces algebraically similar to the classical Volterra operator V acting on $C[0,1]$ are characterized. From this characterization it follows that V does not determine the topology of $C[0,1]$, which answers a question raised by Armando Villena. A sufficient condition for an injective bounded linear operator on a Banach space to determine its topology is obtained. From this condition it follows, for instance, that the Volterra operator acting on the Hardy space $\mathcal{H}^p$ of the unit disk determines the topology of $\mathcal{H}^p$ for any $p \in [1,\infty]$.
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1 Introduction

All vector spaces in this article are assumed to be over the field $\mathbb{C}$ of complex numbers unless specified otherwise. The following problem falls into the family of questions on the so-called automatic continuity. Suppose that T is a bounded linear operator acting on a Banach space X and $\|\cdot\|_1$ is another complete norm on X, with respect to which T is also bounded. Does it follow that $\|\cdot\|_1$ is equivalent to the initial norm? According to the Banach Inverse Mapping Theorem [10], this question is equivalent to the following one. Should $\|\cdot\|_1$ be continuous? If the answer is affirmative, we say that T *determines the topology of* X. One may ask the same question about a family of operators. There are several results both positive and negative for different type of operators and families of operators, see for instance, [1, 5, 7, 8, 9, 13, 14, 15]. The following question was raised by Armando Villena in 2000; it can be found in the Belfast Functional Analysis Day problem book.

Question 1. *Does the Volterra operator*

$$V : C[0,1] \to C[0,1], \qquad Vf(x) = \int_0^x f(t)\,dt$$

determine the topology of $C[0,1]$*?*

We answer this question negatively. To this end bounded linear operators on Banach spaces (algebraically) similar to V are characterized. Two linear operators $T : X \to X$ and $S : Y \to Y$ are said to be *similar* if there exists an invertible linear operator $G : X \to Y$ such that $GT = SG$. Note that we consider similarity **only in the algebraic sense**, that is, even if X and Y carry

some natural topologies, with respect to which T and S are continuous, the operator G **is not assumed to be continuous**.

PROPOSITION **1.1.** *Let T be a bounded linear operator on a Banach space X. Then the following conditions are equivalent.*

(T1) *T determines the topology of X;*

(T2) *if S is a bounded linear operator acting on a Banach space Y, S is similar to T and $G : X \to Y$ is an invertible linear operator such that $GT = SG$, then G is bounded.*

Proof. If T does not determine the topology of X, then there exists a complete norm $\|\cdot\|_0$ on X not equivalent to the initial norm and such that T is bounded with respect to $\|\cdot\|_0$. Let G be the identity operator $Gx = x$ from X endowed with the initial norm $\|\cdot\|_X$ to X endowed with the norm $\|\cdot\|_0$. Since both norms are complete and not equivalent, G is unbounded. If S is the operator T acting on the Banach space $(X, \|\cdot\|_0)$, then the equality $GT = SG$ is obviously satisfied. Thus, (T2) implies (T1).

Suppose now that (T2) is not satisfied. Pick a bounded linear operator S on a Banach space Y and an unbounded invertible linear operator $G : X \to Y$ such that $GT = SG$. Consider the norm on X defined by the formula $\|x\|_0 = \|Gx\|_Y$. Then G is an isometry from $(X, \|\cdot\|_0)$ onto the Banach space Y. Hence X with the norm $\|\cdot\|_0$ is complete. Since G is unbounded, the norm $\|\cdot\|_0$ is not equivalent to the initial one. On the other hand the operator T acting on $(X, \|\cdot\|_0)$ is bounded since it is isometrically similar to S. Thus, T does not determine the topology of X. □

We shall give a negative answer to Question 1 by means of Proposition 1.1, proving that the Volterra operator acting on $C[0,1]$ is similar to a bounded linear operator acting on a Banach space non-isomorphic to $C[0,1]$. Let T be a linear operator acting on a linear space X. The *spectrum* of T is

$$\sigma(T) = \{z \in \mathbb{C} : T - zI \ \text{ is non-invertible}\}.$$

According to the Banach inverse mapping theorem, if X is a Fréchet space and T is continuous, then $\sigma(T)$ coincides with the conventional spectrum since $(T - zI)^{-1}$ is continuous whenever $z \in \mathbb{C} \setminus \sigma(T)$. This is not true for continuous linear operators on general topological vector spaces. We say that T is *quasinilpotent* if $\sigma(T) \subseteq \{0\}$. Everywhere below $\mathbb{Z}$ is the set of integers and $\mathbb{Z}_+$ is the set of non-negative integers.

The following theorem characterizes similarity to the Volterra operator.

THEOREM **1.2.** *Let T be a bounded linear operator on a Banach space X of algebraic dimension $\mathfrak{c} = 2^{\aleph_0}$. Then the following conditions are equivalent.*

(C1) *T is similar to the Volterra operator V acting on $C[0,1]$;*

(C2) *T is injective, quasinilpotent and satisfies the closed finite descent condition, that is, there exists $m \in \mathbb{Z}_+$ for which*

$$\overline{T^m(X)} = \overline{T^{m+1}(X)}. \tag{1}$$

Clearly (1) is satisfied for $m = 0$ if the range of T is dense. Since the algebraic dimension of any separable infinite dimensional Banach space is $\mathfrak{c}$, we have

COROLLARY **1.3.** *Any injective quasinilpotent bounded linear operator T with dense range acting on a separable Banach space is similar to the Volterra operator acting on $C[0,1]$.*

THEOREM **1.4.** *The Volterra operator does not determine the topology of $C[0,1]$.*

Proof. Let V be the Volterra operator acting on $C[0,1]$ and V_2 be the same operator acting on $L_2[0,1]$. By Theorem 1.2 V and V_2 are similar. Since Banach spaces $C[0,1]$ and $L_2[0,1]$ are

not isomorphic, the similarity operator can not be bounded. Proposition 1.1 implies now that V does not determine the topology of $C[0,1]$. $\Box$

We also provide a new sufficient condition for an injective bounded linear operator on a Banach space to determine its topology. It follows that certain injective quasinilpotent operators do determine the topology of the Banach space on which they act. In particular, it is true for the Volterra operator acting on the Hardy space $\mathcal{H}^p$ of the unit disk for any $p \in [1,\infty]$.

THEOREM **1.5.** *Let T be a bounded injective linear operator on a Banach space X such that*

$$\bigcap_{n=0}^{\infty} \overline{T^n(X)} = \{0\} \quad \text{and} \tag{2}$$

$$\text{there exists } n \in \mathbb{Z}_+ \text{ such that } \overline{T^{n+1}(X)} \text{ has finite codimension in } \overline{T^n(X)}. \tag{3}$$

Then T determines the topology of X.

In Section 2 linear operators with empty spectrum are characterized up to similarity. In Section 3 we introduce the class of tame operators and characterize up to similarity tame injective quasinilpotent operators. Section 4 is devoted to auxiliary lemmas with the help of which tame injective bounded operators on Banach spaces are characterized in Section 5. Theorem 1.2 and Theorem 1.5 are proved in Section 6 and Section 7 respectively. In Section 8 of concluding remarks we discuss the previous results and raise few problems.

2 Operators with empty spectrum

In what follows $\mathcal{R}$ stands for the field of complex rational functions considered also as a complex vector space, and

$$M : \mathcal{R} \to \mathcal{R}, \quad Mf(z) = zf(z).$$

LEMMA **2.1.** *Let T be a linear operator on a linear space X such that $\sigma(T) = \varnothing$. Consider the multiplication operation from $\mathcal{R} \times X$ to X defined by the formula $r \cdot x = r(T)x$. Then this multiplication extends the natural multiplication by complex numbers and turns X into a linear space over the field $\mathcal{R}$.*

Proof. Since $\sigma(T) = \varnothing$, $r(T)$ is a well-defined linear operator on X for any $r \in \mathcal{R}$ and $r(T)$ is invertible if $r \neq 0$. Thus, the operation $(r,x) \mapsto r \cdot x$ is well-defined. The verification of axioms of the vector space is fairly elementary. $\Box$

THEOREM **2.2.** *Let T be a linear operator acting on a linear space X such that $\sigma(T) = \varnothing$. Then there exists a unique cardinal $\mu = \mu(T)$ such that T is similar to the direct sum of μ copies of the multiplication operator M. Moreover, $\mu(T)$ coincides with the algebraic dimension of $X_{\mathcal{R}}$, being X, considered as a linear space over $\mathcal{R}$ with the multiplication $r \cdot x = r(T)x$. In particular, two linear operators $T : X \to X$ and $S : Y \to Y$ with $\sigma(T) = \sigma(S) = \varnothing$ are similar if and only if $\mu(T) = \mu(S)$.*

Proof. Let ν be the algebraic dimension of $X_{\mathcal{R}}$ and $\{x_\alpha : \alpha \in A\}$ be a Hamel basis in $X_{\mathcal{R}}$. Then the cardinality of A is ν. For each $\alpha \in A$ let X_α be the $\mathcal{R}$-linear span of the one-element set $\{x_\alpha\}$. Then X_α are linear subspaces of $X_{\mathcal{R}}$ and therefore they are $\mathbb{C}$-linear subspaces of X. Moreover X is the direct sum of X_α. Since for any $x \in X$, the vectors x and Tx are $\mathcal{R}$-collinear, we see that each X_α is T-invariant. Moreover, for any $x \in X$ and any $r \in \mathcal{R}$, $T(r \cdot x) = Mr \cdot x$ and therefore, for any $\alpha \in A$, the restriction T_α of T to X_α is similar to M with the similarity provided by the operator $G_\alpha : \mathcal{R} \to X_\alpha$, $G_\alpha r = r \cdot x_\alpha$. Thus, T is similar to the direct sum of ν copies of M.

Suppose now that μ is a cardinal and T is similar to the direct sum of μ copies of M. Then there exists a set A of cardinality μ and a family $\{X_\alpha : \alpha \in A\}$ of T-invariant linear subspaces of X such that X is the direct sum of X_α and for each $\alpha \in A$, the restriction T_α of T to X_α is similar to M. Let $G_\alpha : \mathcal{R} \to X_\alpha$ be a $\mathbb{C}$-linear invertible linear operator such that $G_\alpha M = T_\alpha G_\alpha$ and $x_\alpha = G_\alpha(1)$. One can easily verify that $G_\alpha(r\rho) = r \cdot G_\alpha(\rho)$ for any $r, \rho \in \mathcal{R}$. Hence for any pairwise different $\alpha_1, \dots, \alpha_n \in A$, the $\mathcal{R}$-linear span of $\{x_{\alpha_1}, \dots, x_{\alpha_n}\}$ coincides with the direct sum of $X_{\alpha_1}, \dots, X_{\alpha_n}$. Therefore $\{x_\alpha : \alpha \in A\}$ is a Hamel basis in $X_{\mathcal{R}}$. Thus, $\mu = \nu$. $\square$

3 Similarity of tame operators

For a linear operator T on a linear space X we denote

$$X_T = \bigcap_{n=0}^{\infty} T^n(X).$$

Clearly X_T is a T-invariant linear subspace of X. Let $T_0 : X_T \to X_T$ be the restriction of T to the invariant subspace X_T.

Lemma **3.1.** *Let T be an injective quasinilpotent operator on a linear space X. Then $\sigma(T_0) = \varnothing$.*

Proof. Let $z \in \mathbb{C}$. We have to prove that $T_0 - zI$ is invertible. Since $T_0 - zI$ is the restriction of the injective operator $T - zI$, we see that $T_0 - zI$ is injective. It remains to verify surjectivity of $T_0 - zI$. Let $x_0 \in X_T$. Since T is injective and $x_0 \in X_T$, for any $n \in \mathbb{Z}_+$ there exists a unique $x_n \in X$ such that $T^n x_n = x_0$. From injectivity of T it also follows that $x_n = T^m x_{m+n}$ for each $m, n \in \mathbb{Z}_+$. Therefore all x_n belong to X_T.

Case $z = 0$. Since $x_1 \in X_0$ and $Tx_1 = T_0 x_1 = x_0$, we see that $T_0 = T_0 - zI$ is surjective.

Case $z \neq 0$. Then $T - zI$ is invertible. Denote $y = (T - zI)^{-1} x_0$. Then for any $n \in \mathbb{Z}_+$,

$$y = (T - zI)^{-1} T^n x_n = T^n w_n, \quad \text{where} \quad w_n = (T - zI)^{-1} x_n.$$

Hence $y \in X_T$ and $(T_0 - zI)y = x_0$. Thus, $T_0 - zI$ is surjective. $\square$

In order to formulate the main result of this section we need some additional notation. If X is a linear space, E is a linear subspace of X and $x, y \in X$, we write

$$x \equiv y \ (\mathrm{mod}\, E)$$

if $x - y \in E$. We also say that a family $\{x_\alpha\}_{\alpha \in A}$ of elements of X is *linearly independent modulo* E if for any pairwise different $\alpha_1, \dots, \alpha_n \in A$ and any complex numbers $c_1, \dots, c_n$, the inclusion $c_1 x_{\alpha_1} + \ldots + c_n x_{\alpha_n} \in E$ implies $c_j = 0$ for $1 \leqslant j \leqslant n$. Clearly linear independence of $\{x_\alpha\}_{\alpha \in A}$ modulo E is equivalent to linear independence of $\{\pi(x_\alpha)\}_{\alpha \in A}$ in X/E, where $\pi : X \to X/E$ is the canonical map.

Definition **1.** We say that a linear operator T acting on a linear space X is *tame* if for any sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ of elements of X there exists $x \in X$ such that for each $n \in \mathbb{Z}_+$,

$$x \equiv \sum_{k=0}^{n} T^k x_k \ (\mathrm{mod}\, T^{n+1}(X)). \tag{4}$$

Obviously surjective operators are tame and tameness is invariant under similarities. It is not clear *a priori* whether there are injective tame operators T with $\sigma(T) = \{0\}$. We shall show in the next section that the Volterra operator acting on $C[0,1]$ is tame, which motivates the study

of similarity of injective quasinilpotent tame operators. In what follows $\dim_{\mathbb{K}} X$ stands for the algebraic dimension of the linear space X over the field $\mathbb{K}$.

Theorem **3.2.** *Let $T : X \to X$ and $S : Y \to Y$ be two injective quasinilpotent tame operators. Then T and S are similar if and only if $\dim_{\mathbb{C}} X/T(X) = \dim_{\mathbb{C}} Y/T(Y)$ and $\dim_{\mathcal{R}} X_T = \dim_{\mathcal{R}} Y_S$, where the multiplication of $x \in X_T$ and $y \in Y_S$ by $r \in \mathcal{R}$ are given by $r \cdot x = r(T_0)x$ and $r \cdot y = r(S_0)y$ respectively.*

The rest of the section is devoted to the proof of Theorem 3.2. We need some preparation.

3.1 Auxiliary Lemmas

We set

$$\mathcal{R}_+ = \{f \in \mathcal{R} : f(0) \neq \infty\}.$$

If T is injective and quasinilpotent, then $r(T)$ is a well-defined injective linear operator on X for any $r \in \mathcal{R}_+ \setminus \{0\}$ and $r(T)$ is invertible if $r(0) \in \mathbb{C} \setminus \{0\}$.

Definition **2.** Let T be an injective quasinilpotent operator acting on a linear space X. We say that a family of vectors $\{x_\alpha\}_{\alpha \in A}$ in X is *T-independent* if for any pairwise different $\alpha_1, \dots, \alpha_n \in A$ and any $r_1, \dots, r_n \in \mathcal{R}_+$, the inclusion $r_1(T)x_{\alpha_1} + \ldots + r_n(T)x_{\alpha_n} \in X_T$ implies $r_j = 0$ for $1 \leqslant j \leqslant n$.

From Lemma 3.1 it follows that if $T : X \to X$ is injective and quasinilpotent, then $r(T)(X_T) = X_T$ for any $r \in \mathcal{R}_+ \setminus \{0\}$. From this observation it follows that a family $\{x_\alpha\}_{\alpha \in A}$ in X is T-*independent* if and only if for any pairwise different $\alpha_1, \dots, \alpha_n \in A$ and any polynomials $p_1, \dots, p_n$ the inclusion $p_1(T)x_{\alpha_1} + \ldots + p_n(T)x_{\alpha_n} \in X_T$ implies $p_j = 0$ for $1 \leqslant j \leqslant n$. Applying the Zorn Lemma to the set of T-independent families partially ordered by inclusion, we see that there are maximal T-independent families.

Lemma **3.3.** *Let T be an injective quasinilpotent operator acting on a linear space X and $\{x_\alpha\}_{\alpha \in A}$ be a maximal T-independent family. Then for any $x \in X$ there exists a unique finite (maybe empty) subset $\Lambda = \Lambda(x)$ of A such that*

$$T^n x = w_n + \sum_{\alpha \in \Lambda} r_{\alpha,n}(T) x_\alpha \tag{5}$$

for some $n \in \mathbb{Z}_+$, $w_n \in X_T$ and non-zero $r_{\alpha,n} \in \mathcal{R}_+$. Moreover w_n and $r_{\alpha,n}$ are uniquely determined by n and x. Finally, if the decomposition (5) does exist for some n then it exists for all greater n and $w_m = T^{m-n} w_n$, $r_{\alpha,m} = M^{m-n} r_{\alpha,n}$ for $m \geqslant n$.

Proof. Let B and C be finite subsets of A, r_α for $\alpha \in B$ and ρ_α for $\alpha \in C$ be non-zero elements of $\mathcal{R}_+$, $m, n \in \mathbb{Z}_+$ and $u, v \in X_T$ be such that

$$T^n x = u + \sum_{\alpha \in B} r_\alpha(T) x_\alpha \quad \text{and} \quad T^m x = v + \sum_{\alpha \in C} \rho_\alpha(T) x_\alpha.$$

Let also $D = B \cup C$ and r'_α, $\rho'_\alpha \in \mathcal{R}_+$ for $\alpha \in D$ be defined by the formulas: $r'_\alpha = M^m r_\alpha$ for $\alpha \in B$, $\rho'_\alpha = M^n \rho_\alpha$ for $\alpha \in C$, $r'_\alpha = 0$ for $\alpha \in C \setminus B$ and $\rho'_\alpha = 0$ for $\alpha \in B \setminus C$. Then

$$T^{m+n} x = T^m u + \sum_{\alpha \in D} r'_\alpha(T) x_\alpha = T^n v + \sum_{\alpha \in D} \rho'_\alpha(T) x_\alpha. \tag{6}$$

Hence

$$\sum_{\alpha \in D} (r'_\alpha - \rho'_\alpha)(T) x_\alpha \in X_T.$$

Since $\{x_\alpha\}_{\alpha\in A}$ is T-*independent*, we have $\rho'_\alpha = r'_\alpha$ for each $\alpha \in D$. Hence $B = C = D$ and $M^m r_\alpha = M^n \rho_\alpha$ for any $\alpha \in D$. Substituting these equalities into (6), we see that $T^m u = T^n v$. This means that the set Λ is uniquely determined by x and w_n, $r_{\alpha,n}$ are uniquely determined by x and n. Moreover, if (5) is satisfied then it is satisfied for greater n with $w_m = T^{m-n} w_n$ and $r_{\alpha,m} = M^{m-n} r_{\alpha,n}$ for $m \geqslant n$.

Suppose now that $x \in X$ does not admit a decomposition (5). It easily follows that $\{x\} \cup \{x_\alpha\}_{\alpha\in A}$ is T-independent, which contradicts the maximality of $\{x_\alpha\}_{\alpha\in A}$. □

LEMMA **3.4.** *Let T be an injective linear operator acting on a linear space X and E be a linear subspace of X such that $E \oplus T(X) = X$. Then for any $x \in X$ there exists a unique sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ of elements of E such that* (4) *is satisfied for any $n \in \mathbb{Z}_+$.*

Proof. Let $x \in X$. It suffices to prove that for any $m \in \mathbb{Z}_+$, there exist unique $x_0, \dots, x_m \in E$ for which (4) is satisfied for $n \leqslant m$. We achieve this using induction with respect to m.

Since $E \oplus T(X) = X$, there exists a unique $x_0 \in E$ for which $x - x_0 \in T(X)$. This inclusion is exactly (4) for $n = 0$. The basis of induction is constructed. Let now m be a positive integer. Assume that there exist unique $x_0, \dots, x_{m-1} \in E$ for which (4) is satisfied for $n \leqslant m-1$. Since T is injective, (4) for $n = m-1$ implies that there exists a unique $u \in X$ such that $x - x_0 - Tx_1 - \dots - Tx_{m-1} = T^m u$. Since $E \oplus T(X) = X$, there exists a unique $x_m \in E$ for which $u - x_m \in T(X)$. Since the last inclusion is equivalent to (4) for $n = m$, the induction step is complete and so is the proof of the lemma. □

3.2 Proof of Theorem 3.2

Let T and S be similar. Since the algebraic codimension of the range of a linear operator is a similarity invariant, we have $\dim_{\mathbb{C}} X/T(X) = \dim_{\mathbb{C}} Y/T(Y)$. Let $G : X \to Y$ be an invertible linear operator such that $GT = SG$. Clearly $G(X_T) = Y_S$. Therefore T_0 and S_0 are similar. According to Lemma 3.1 and Theorem 2.2 $\dim_{\mathcal{R}} X_T = \dim_{\mathcal{R}} Y_S$.

Suppose now that $\dim_{\mathbb{C}} X/T(X) = \dim_{\mathbb{C}} Y/T(Y)$ and $\dim_{\mathcal{R}} X_T = \dim_{\mathcal{R}} Y_S$. By Lemma 3.1 and Theorem 2.2, T_0 and S_0 are similar. Hence there exists an invertible linear operator $G_0 : X_T \to Y_S$ such that

$$G_0 T x = S G_0 x \ \text{ for any } \ x \in X_T. \tag{7}$$

Pick linear subspaces E and F in X and Y respectively such that $E \oplus T(X) = X$ and $F \oplus T(Y) = Y$. Since $\dim_{\mathbb{C}} X/T(X) = \dim_{\mathbb{C}} Y/T(Y)$, we have $\dim_{\mathbb{C}} E = \dim_{\mathbb{C}} F$ and therefore there exists an invertible linear operator $G_1 : E \to F$. Let $\{x_\alpha\}_{\alpha\in A}$ be a maximal T-independent family in X. According to Lemma 3.4, for any $\alpha \in A$, there exists a unique sequence $\{x_{\alpha,n}\}_{n\in\mathbb{Z}_+}$ of elements of E such that

$$x_\alpha \equiv \sum_{j=0}^{n} T^j x_{\alpha,j} \ (\operatorname{mod} T^{n+1}(X)) \ \text{ for each } \ n \in \mathbb{Z}_+. \tag{8}$$

Since S is tame, for any $\alpha \in A$, there exists $y_\alpha \in Y$ such that

$$y_\alpha \equiv \sum_{j=0}^{n} S^j G_1 x_{\alpha,j} \ (\operatorname{mod} S^{n+1}(Y)) \ \text{ for each } \ n \in \mathbb{Z}_+. \tag{9}$$

Let $x \in X$ and $\Lambda = \Lambda(x)$ be the finite subset of A furnished by Lemma 3.3. Pick a positive integer n, $w_n \in X_T$ and non-zero $r_{\alpha,n} \in \mathcal{R}_+$ for $\alpha \in \Lambda$ such that (5) is satisfied. Let

$$y_n = G_0 w_n + \sum_{\alpha\in\Lambda} r_{\alpha,n}(S) y_\alpha. \tag{10}$$

First, let us verify that $y_n \in S^n(Y)$. According to the classical Taylor theorem, for any $\alpha \in \Lambda$, there exists a unique polynomial

$$p_{\alpha,n}(z) = \sum_{j=0}^{n-1} a_{\alpha,j} z^j$$

such that $r_{\alpha,n} - p_{\alpha,n}$ has zero of order at least n in zero. Then $r_{\alpha,n}(T)x_\alpha \equiv p_{\alpha,n}(T)x_\alpha \pmod{T^n(X)}$. Using (8), we have

$$r_{\alpha,n}(T)x_\alpha \equiv p_{\alpha,n}(T)x_\alpha \equiv \sum_{j=0}^{n-1} T^j \left(\sum_{k=0}^{j} a_{\alpha,k} x_{\alpha,j-k} \right) \pmod{T^n(X)}.$$

Summing over α, we obtain

$$\sum_{\alpha\in\Lambda} r_{\alpha,n}(T)x_\alpha \equiv \sum_{j=0}^{n-1} T^j \left(\sum_{\alpha\in\Lambda} \sum_{k=0}^{j} a_{\alpha,k} x_{\alpha,j-k} \right) \pmod{T^n(X)}. \tag{11}$$

Similarly using (9) instead of (8), we have

$$\sum_{\alpha\in\Lambda} r_{\alpha,n}(S)y_\alpha \equiv \sum_{j=0}^{n-1} S^j G_1 \left(\sum_{\alpha\in\Lambda} \sum_{k=0}^{j} a_{\alpha,k} x_{\alpha,j-k} \right) \pmod{S^n(Y)}. \tag{12}$$

From (5) and (11) it follows that

$$\sum_{j=0}^{n-1} T^j \left(\sum_{\alpha\in\Lambda} \sum_{k=0}^{j} a_{\alpha,k} x_{\alpha,j-k} \right) \equiv 0 \pmod{T^n(X)}.$$

According to the uniqueness part of Lemma 3.4,

$$\sum_{\alpha\in\Lambda} \sum_{k=0}^{j} a_{\alpha,k} x_{\alpha,j-k} = 0 \quad \text{for } 0 \leqslant j \leqslant n-1.$$

Substituting this into (12) and using (10), we obtain $y_n \equiv 0 \pmod{S^n(Y)}$, or equivalently, $y_n \in S^n(Y)$. Thus, there exists a unique $y \in Y$ for which $S^n y = y_n$. We write $Gx = y$.

Let us check that the map G is well-defined. To this end it suffices to verify that y does not depend on n. Let $m > n$ and the vectors y_n and y_m be defined by (10). According to Lemma 3.3, $u_m = T^{m-n}u_n$ and $r_{\alpha,m} = M^{m-n}r_{\alpha,n}$ for each $\alpha \in \Lambda$. Substituting these equalities into (10) and taking (7) into account, we see that $y_m = S^{m-n}y_n$ and therefore y does not depend on the choice of n. The map $G : X \to Y$ is defined. The linearity of G follows easily from its definition. Thus, we have a linear operator $G : X \to Y$.

First, let us show that $GT = SG$. Let $x \in X$, Λ be the finite set furnished by Lemma 3.3 and $n \in \mathbb{Z}_+$, $u_n \in X_T$, $r_{\alpha,n} \in \mathcal{R}_+ \setminus \{0\}$ for $\alpha \in \Lambda$ be such that (5) is satisfied. By definition of G, we have $Gx = y$, where $S^n y = y_n$ and y_n is defined in (10). Then $S^{n-1}SGx = y_n$. By (5),

$$T^{n-1}Tx = w_n + \sum_{\alpha\in\Lambda} r_{\alpha,n}(T)x_\alpha.$$

From the definition of G and (10) it follows that $S^{n-1}GTx = y_n$. Hence $S^{n-1}GTx = S^{n-1}SGx$. Since S is injective, we have $GTx = SGx$. Thus, $GT = SG$.

Next, we shall verify the injectivity of G. Suppose that $x \in X$ and $Gx = 0$. First, let us consider the case $x \notin X_{T}$. In this case the finite subset $\Lambda = \Lambda(x)$ of A, provided by Lemma 3.3, is non-empty. Let $n \in \mathbb{Z}_+$, $u_n \in X_{T}$, $r_{\alpha,n} \in \mathcal{R}_+ \setminus \{0\}$ for $\alpha \in \Lambda$ be such that (5) is satisfied. Since $x \notin X_{T}$, there exists a positive integer m such that $T^n x \notin T^m(X)$. Choose polynomials

$$p_{\alpha,m}(z) = \sum_{j=0}^{m-1} a_{\alpha,j} z^j$$

such that $r_{\alpha,n} - p_{\alpha,m}$ has zero of order at least m in zero. As above, we have

$$T^n x \equiv \sum_{\alpha\in\Lambda} r_{\alpha,n}(T)x_\alpha \equiv \sum_{j=0}^{m-1} T^j \biggl(\sum_{\alpha\in\Lambda}\sum_{k=0}^{j} a_{\alpha,k}x_{\alpha,j-k}\biggr) \ (\mathrm{mod}\, T^m(X)) \ \text{ and} \tag{13}$$

$$S^n Gx \equiv \sum_{\alpha\in\Lambda} r_{\alpha,n}(S)y_\alpha \equiv \sum_{j=0}^{m-1} S^j G_1 \biggl(\sum_{\alpha\in\Lambda}\sum_{k=0}^{j} a_{\alpha,k}x_{\alpha,j-k}\biggr) \ (\mathrm{mod}\, S^m(Y)). \tag{14}$$

Since $T^n x \notin T^m(X)$, (13) implies that there exists j, $0 \leqslant j \leqslant m-1$ such that

$$\sum_{\alpha\in\Lambda}\sum_{k=0}^{j} a_{\alpha,k}x_{\alpha,j-k} \neq 0.$$

Since G_1 is injective,

$$G_1 \sum_{\alpha\in\Lambda}\sum_{k=0}^{j} a_{\alpha,k}x_{\alpha,j-k} \neq 0.$$

Using (14) and the last display, we have $S^n Gx \notin S^m(Y)$. Therefore $Gx \neq 0$. This contradiction shows that $x \in X_{T}$. Since the restriction of G to X_{T} coincides with G_0 and G_0 is injective, we obtain $x = 0$. Injectivity of G is proven.

Finally let us prove that G is onto. Since the restriction of G to X_{T} coincides with G_0 and the restriction of G to E coincides with G_1, from the definition of G and the equality $GT = SG$ it follows that

$$\begin{array}{l}\text{if } x \in X \text{ and the sequence } \{x_n\}_{n\in\mathbb{Z}_+} \text{ of elements of } E \text{ satisfy (4) for any } n \in \mathbb{Z}_+,\\ \text{then } Gx \equiv G_1x_0 + \ldots + S^nG_1x_n \ (\mathrm{mod}\, S^{n+1}(Y)) \text{ for each } n \in \mathbb{Z}_+.\end{array} \tag{15}$$

Let $y \in Y$. According to Lemma 3.4 there exists a sequence $\{y_n\}_{n\in\mathbb{Z}_+}$ of elements of F for which $y \equiv y_0 + \ldots + S^n y_n \ (\mathrm{mod}\, S^{n+1}(Y))$ for any $n \in \mathbb{Z}_+$. Since T is tame, there exists $x \in X$ such that (4) is satisfied for any $n \in \mathbb{Z}_+$ with $x_n = G_1^{-1} y_n$. Using (15), we see that

$$Gx \equiv y \equiv y_0 + \ldots + S^n y_n \ (\mathrm{mod}\, S^{n+1}(Y)) \ \text{ for any } n \in \mathbb{Z}_+.$$

Hence $Gx - y \in Y_{S}$. Therefore $G(x - G_0^{-1}(Gx - y)) = Gx - Gx + y = y$ and y is in the range of G. Surjectivity of G is proven. Thus, $G : X \to Y$ is an invertible linear operator and $GT = SG$ and therefore T and S are similar. The proof of Theorem 3.2 is complete.

4 Auxiliary results

Recall that a Fréchet space is a complete metrizable locally convex topological vector space. We shall prove several lemmas, which will be used in the proof of Theorems 1.2 and 1.5.

4.1 Algebraic dimensions

LEMMA **4.1.** *Let $\{a_n\}_{n\in\mathbb{Z}_+}$ be a sequence of non-negative numbers such that the set $\{n\in\mathbb{Z}_+ : a_n>0\}$ is infinite and $a_n^{1/n}\to 0$ as $n\to\infty$. Then there exists an infinite set $A\subseteq\mathbb{Z}_+$ such that $a_n>0$ for each $n\in A$ and $\sum\limits_{k=n+1}^{\infty} a_k = o(a_n)$ as $n\to\infty$, $n\in A$.*

Proof. For $\alpha>0$ denote $c_\alpha=\sup\limits_{n\in\mathbb{Z}_+} a_n\alpha^{-n}$. Since $a_n^{1/n}\to 0$ as $n\to\infty$ and there are positive a_n, we see that for any $\alpha>0$, $c_\alpha\in(0,\infty)$ and there is $n(\alpha)\in\mathbb{Z}_+$ for which $a_{n(\alpha)}=c_\alpha\alpha^{n(\alpha)}$.

Suppose now that the required set A does not exist. Then there exists $c>0$ such that $\sum\limits_{k=n+1}^{\infty} a_k\geqslant ca_n$ for each $n\in\mathbb{Z}_+$. Since $a_n\leqslant c_\alpha\alpha^n$ for any $\alpha>0$ and any $n\in\mathbb{Z}_+$, we have

$$ca_{n(\alpha)}=cc_\alpha\alpha^{n(\alpha)}\leqslant\sum_{k=n(\alpha)+1}^{\infty}a_k\leqslant\sum_{k=n(\alpha)+1}^{\infty}c_\alpha\alpha^{k+1}=\frac{\alpha c_\alpha}{1-\alpha}\alpha^{n(\alpha)}$$

for any $\alpha\in(0,1)$. Hence $c\leqslant\alpha/(1-\alpha)$ for each $\alpha\in(0,1)$, which is impossible since c is positive. This contradiction completes the proof. □

LEMMA **4.2.** *Let X be a Fréchet space and $\{x_n\}_{n\in\mathbb{Z}_+}$ be a linearly independent sequence in X. Then there exists a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that for any sequence $\{b_n\}_{n\in\mathbb{Z}_+}$ of complex numbers,*

$$\text{if } \sum_{n=0}^{\infty}t_n|b_n|<\infty \text{ then the series } \sum_{n=0}^{\infty}b_nx_n \text{ is absolutely convergent in } X; \tag{16}$$

$$\text{if } \sum_{n=0}^{\infty}t_n|b_n|<\infty \text{ and } \sum_{n=0}^{\infty}b_nx_n=0 \text{ then } b_n=0 \text{ for each } n\in\mathbb{Z}_+. \tag{17}$$

Proof. Let $\{p_n\}_{n\in\mathbb{Z}_+}$ be a sequence of seminorms defining the topology of X such that $p_{n+1}(x)\geqslant p_n(x)$ for any $x\in X$ and $n\in\mathbb{Z}_+$. Since x_n are linearly independent, we can without loss of generality, assume that

$$\inf\{p_n(x_n-y): y\in \operatorname{span}\{x_1,\dots,x_{n-1}\}\}\geqslant\varepsilon_n\in(0,1]; \tag{18}$$

$$p_n(x_n)=1 \text{ for all } n\in\mathbb{Z}_+. \tag{19}$$

Indeed, if it is not the case, we can replace p_n by a subsequence p_{k_n} to make (18) valid and then take $x_n/p_n(x_n)$ instead of x_n to make (19) valid.

Evidently, there exists an increasing sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that

$$\sum_{n=0}^{\infty}\frac{p_k(x_n)}{t_n}<+\infty \quad \text{for any} \quad k\in\mathbb{N}; \tag{20}$$

$$\lim_{n\to\infty}\frac{\varepsilon_nt_{n+1}}{t_n}=+\infty. \tag{21}$$

We shall show that the sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ has all desired properties. The condition (16) follows from (20). Let us prove (17). Suppose that $\sum\limits_{n=0}^{\infty}t_n|b_n|<\infty$, $\sum\limits_{n=0}^{\infty}b_nx_n=0$ and there exists $m\in\mathbb{Z}_+$ for which $b_m\neq 0$. Since x_n are linearly independent, there are infinitely many $m\in\mathbb{N}$ such that

$b_m \neq 0$. From (21) and the inequality $\sum\limits_{n=0}^{\infty} t_n|b_n| < \infty$ it follows that $\lim\limits_{n\to\infty}\Bigl(b_n \prod\limits_{k=0}^{n-1} \varepsilon_k^{-1}\Bigr)^{1/n} = 0$. Applying Lemma 4.1 to the sequence $a_0 = |b_0|$, $a_n = |b_n|\varepsilon_0^{-1}\cdot\ldots\cdot\varepsilon_{n-1}^{-1}$ for $n \geqslant 1$, we see that there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers such that $b_{n_k} \neq 0$ for each $k \in \mathbb{Z}_+$ and $\sum\limits_{m=n_k+1}^{\infty} |b_m| = o(\varepsilon_{n_k} b_{n_k})$ as $k \to \infty$. Thus, there exists $r \in \mathbb{Z}_+$ for which $\sum\limits_{n=r+1}^{\infty} |b_n| < \frac{\varepsilon_r|b_r|}{2}$. Then

$$0 = p_r\Bigl(\sum_{n=0}^{\infty} b_n x_n\Bigr) \geqslant |b_r| \inf\{p_r(x_r - y) : y \in \operatorname{span}\{x_1,\ldots,x_{r-1}\}\} - \sum_{n=r+1}^{\infty} |b_n| p_r(x_n) \geqslant$$
$$\geqslant |b_r|\varepsilon_r - \sum_{n=r+1}^{\infty} |b_n| p_n(x_n) \geqslant |b_r|\varepsilon_r - \sum_{n=r+1}^{\infty} |b_n| > \frac{|b_r|\varepsilon_r}{2} > 0.$$

This contradiction proves (17). $\square$

Lemma 4.2 gives an alternative proof of the following well-known fact. The classical proof goes along the same lines as the proof of a more general Lemma 4.5 below.

COROLLARY **4.3.** *Let X be an infinite dimensional Fréchet space. Then* $\dim_{\mathbb{C}} X \geqslant \mathfrak{c}$.

Proof. Since X is infinite dimensional, there exists a linearly independent sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ in X. By Lemma 4.2 there exists a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ of positive numbers, satisfying (16) and (17). Pick a sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that $\sum\limits_{n=0}^{\infty} c_n t_n < \infty$. For any z from the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ consider $y_z = \sum\limits_{n=0}^{\infty} c_n z^n x_n$. The series converges absolutely in X according to (16). Since the sequences $s_z = \{c_n z^n\}_{n\in\mathbb{Z}_+}$ are linearly independent, (17) implies that the family of vectors $\{y_z\}_{z\in\mathbb{T}}$ is linearly independent in X. Since $\mathbb{T}$ has cardinality $\mathfrak{c}$, we have $\dim_{\mathbb{C}} X \geqslant \mathfrak{c}$. $\square$

The next proposition deals with continuous linear operators with empty spectrum acting on Fréchet spaces. It worth noting that such operators do exist. For instance, let X be the space of infinitely differentiable functions $f : [0,1] \to \mathbb{C}$ such that $f^{(j)}(0) = 0$ for any $j \in \mathbb{Z}_+$ endowed with the topology of uniform convergence of all derivatives. Then X is a Fréchet space and the Volterra operator acts continuously on X and has empty spectrum.

PROPOSITION **4.4.** *Let X be a non-zero Fréchet space, $T : X \to X$ be a continuous linear operator with empty spectrum and $X_{\mathcal{R}}$ be X considered as a linear space over $\mathcal{R}$ with the multiplication $r \cdot x = r(T)x$. Then* $\dim_{\mathbb{C}} X = \dim_{\mathcal{R}} X_{\mathcal{R}}$.

Proof. Let $\nu = \dim_{\mathbb{C}} X$ and $\mu = \dim_{\mathcal{R}} X_{\mathcal{R}}$. Since there is no operators with empty spectrum on a finite dimensional space, we see that X is infinite dimensional. By Corollary 4.3, $\nu \geqslant \mathfrak{c}$. Taking into account that the algebraic dimension of $\mathcal{R}$ as a linear space over $\mathbb{C}$ equals $\mathfrak{c}$, we see that $\mu \cdot \mathfrak{c} = \nu$. Hence $\mu \leqslant \nu$ and $\mu = \nu$ if $\nu > \mathfrak{c}$. It remains to verify that $\mu \geqslant \mathfrak{c}$.

Let $x \in X \setminus \{0\}$. Since $\sigma(T) = \varnothing$, the sequence $\{T^n x\}_{n\in\mathbb{Z}_+}$ is linearly independent. By Lemma 4.2 there exists a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ of positive numbers, satisfying (16) and (17) for $x_n = T^n x$. Pick a sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that $\sum\limits_{n=m}^{\infty} c_{n-m} t_n < \infty$ for each $m \in \mathbb{Z}_+$ and $\lim\limits_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$.

Recall that $A \subset \mathbb{T}$ is called independent if $z_1^{k_1}\cdot\ldots\cdot z_n^{k_n} \neq 1$ for any pairwise different $z_1,\ldots,z_n \in A$ and any non-zero integers $k_1,\ldots,k_n$. Let $A \subset \mathbb{T}$ be an independent set of cardinality $\mathfrak{c}$ (such a set can even be chosen to be compact, see [6]). According to the classical Kronecker theorem [6]

for any pairwise different $z_1,\dots,z_n\in A$, the sequence $\{(z_1^k,\dots,z_n^k)\}_{k\in\mathbb{Z}_+}$ is dense in $\mathbb{T}^n$. Therefore for any pairwise different $z_1,\dots,z_n\in A$ and any $w_1,\dots,w_n\in\mathbb{C}$,

$$\varlimsup_{k\to\infty}\Bigl|\sum_{j=1}^n w_j z_j^k\Bigr| = \sum_{j=1}^n |w_j|. \tag{22}$$

For each $z\in A$ consider the vector $y_z=\sum\limits_{n=0}^{\infty} c_n z^n T^n x$. The series converges absolutely in X according to (16) for $x_n=T^n x$. In order to prove that $\mu\geqslant\mathfrak{c}$ it suffices to verify that the family $\{y_z\}_{z\in A}$ is linearly independent in $X_{\mathcal{R}}$.

Suppose the contrary. Then there exist pairwise different $z_1,\dots,z_n\in A$ and $r_1,\dots r_n\in\mathcal{R}\backslash\{0\}$ such that $r_1(T)y_{z_1}+\ldots+r_n(T)y_{z_n}=0$. Multiplying r_j by the least common multiple of their denominators, we see that there are non-zero polynomials $p_1,\dots,p_n$ such that $p_1(T)y_{z_1}+\ldots+p_n(T)y_{z_n}=0$. Let $m=\max\limits_{1\leqslant j\leqslant n}\deg p_j$. Then

$$p_j(z)=\sum_{l=0}^m a_{j,l}z^l \ \text{ for } 1\leqslant j\leqslant n, \ \text{ where } a_{j,l}\in\mathbb{C} \text{ and } \sum_{j=1}^n |a_{j,m}|>0. \tag{23}$$

From the definition of y_z and continuity of T it follows that

$$0=\sum_{j=1}^n p_j(T)y_{z_j}=\sum_{k=0}^\infty \alpha_k T^k x, \ \text{ where } \alpha_k=\sum_{j=1}^n\sum_{l=0}^{\min\{m,k\}} a_{j,l}z_j^{k-l}c_{k-l}.$$

For $k\geqslant m$, we have

$$\alpha_k=\beta_k+\gamma_k, \ \text{ where } \beta_k=c_{k-m}\sum_{j=1}^n a_{j,m}z_j^{k-m} \text{ and } \gamma_k=\sum_{l=0}^m c_{k-l}\sum_{j=1}^n a_{j,l}z_j^{k-l}.$$

Clearly $\gamma_k=O(c_{k-m+1})$ as $k\to\infty$. Using (22) and (23), we have

$$\varlimsup_{k\to\infty}\frac{|\beta_k|}{c_{k-m}}=\sum_{j=1}^n |a_{j,m}|>0.$$

Taking into account that $c_{k-m+1}=o(c_{k-m})$ as $k\to\infty$, we see that there is an infinite set $\Lambda\subset\mathbb{Z}_+$ such that $\beta_k\neq 0$ for any $k\in\Lambda$ and $\gamma_k/\beta_k\to 0$ as $k\to\infty$, $k\in\Lambda$. It follows that $\alpha_k=\beta_k+\gamma_k\neq 0$ for sufficiently large $k\in\Lambda$. Since $\alpha_k=O(c_{k-m})$ as $k\to\infty$ and $\sum\limits_{k=m}^\infty c_{k-m}t_k<\infty$, we have $\alpha_k=0$ for each $k\in\mathbb{Z}_+$ according to (17) for $x_n=T^n x$. This contradiction proves the linear independence of $\{y_z\}_{z\in A}$ in $X_{\mathcal{R}}$. $\square$

Lemma 4.5. *Let X be a complete metrizable topological vector space, Y be a linear subspace of X, which is a union of countably many closed subsets of X. Suppose also that X/Y is infinite dimensional. Then $\dim_{\mathbb{C}}X/Y\geqslant\mathfrak{c}$.*

Proof. Pick a sequence $\{B_n\}_{n\in\mathbb{Z}_+}$ of closed subsets of X such that $B_n\subseteq B_{n+1}$ for any $n\in\mathbb{Z}_+$ and $Y=\bigcup\limits_{n=0}^\infty B_n$. Since X is a complete metrizable topological vector space there exists a complete metric d on X defining the topology of X. Indeed, the topology of any metrizable topological vector space can be defined by a shift-invariant metric d, that is $d(x,y)=d(x+u,y+u)$ for any x,

y and u and any shift-invariant metric, defining the topology of a complete metrizable topological vector space is complete, see [11]. Recall that the diameter of a subset A of a metric space (X,d) is

$$\operatorname{diam} A = \sup_{x,y\in A} d(x,y).$$

Let

$$\mathbb{D} = \bigcup_{n=1}^{\infty} \mathbb{D}_n, \quad \text{where} \quad \mathbb{D}_n = \{\varepsilon = (\varepsilon_0,\dots,\varepsilon_n) : \varepsilon_j \in \{0,1\} \ \text{ for } \ 0 \leqslant j \leqslant n\}.$$

We shall construct a family $\{U_\alpha\}_{\alpha\in\mathbb{D}}$ of open subsets of X such that for any $n\in\mathbb{Z}_+$,

(S1) $\overline{U}_\alpha \subset U_{\alpha_1,\dots,\alpha_{n-1}}$ for any $\alpha\in\mathbb{D}_n$ if $n\geqslant 1$;

(S2) $\sum\limits_{\alpha\in\mathbb{D}_n} z_\alpha y_\alpha \notin B_n$ for any $y_\alpha\in U_\alpha$ and any complex numbers z_α such that $\sum\limits_{\alpha\in\mathbb{D}_n} |z_\alpha| = 1$;

(S3) $\operatorname{diam} U_\alpha \leqslant 2^{-n}$ for any $\alpha\in\mathbb{D}_n$.

On step 0 pick $x_0,x_1\in X$ linearly independent modulo Y. Since $B_0\subseteq Y$ is closed in X, we can choose neighborhoods U_0 and U_1 of x_0 and x_1 respectively small enough to ensure that, $\operatorname{diam} U_0\leqslant 1$, $\operatorname{diam} U_1\leqslant 1$ and $z_0y_0+z_1y_1\notin B_0$ if $y_0\in\overline{U}_0$, $y_1\in\overline{U}_1$ and $|z_0|+|z_1|=1$. Thus, conditions (S1–S3) for $n=0$ are satisfied.

Suppose now that m is a positive integer and U_α for $\alpha\in\bigcup\limits_{k=0}^{m-1}\mathbb{D}_k$, satisfying (S1–S3) for $n<m$ are already constructed. Since Y has infinite codimension in X, there exist $x_\alpha\in X$ for $\alpha\in\mathbb{D}_m$ such that $x_\alpha\in U_{(\alpha_1,\dots,\alpha_{m-1})}$ for each $\alpha\in\mathbb{D}_m$ and the vectors $\{x_\alpha\}_{\alpha\in\mathbb{D}_m}$ are linearly independent modulo Y (we use the obvious fact that the linear span of any non-empty open subset of a topological vector space is the whole space). Since $B_m\subseteq Y$ is closed in X, we can choose neighborhoods U_α of x_α for $\alpha\in\mathbb{D}_m$ small enough to ensure that (S1–S3) for $n=m$ are satisfied. The induction procedure is complete.

Conditions (S1), (S3) and completeness of (X,d) imply that for any $\alpha\in\{0,1\}^{\mathbb{Z}_+}$, $\bigcap\limits_{n=1}^{\infty} U_{(\alpha_1,\dots,\alpha_n)}$ is a one-element set $\{u_\alpha\}$. Since the cardinality of the set $\{0,1\}^{\mathbb{Z}_+}$ is $\mathfrak{c}$, it suffices to verify that the family $\{u_\alpha\}_{\alpha\in\{0,1\}^{\mathbb{Z}_+}}$ is linearly independent modulo Y. Suppose the contrary. Then there exist pairwise different $\alpha^1,\dots,\alpha^m\in\{0,1\}^{\mathbb{Z}_+}$ and complex numbers $z_1,\dots,z_m$ such that

$$\sum_{j=1}^{m} z_j u_{\alpha^j}\in Y \quad\text{and}\quad \sum_{j=1}^{m}|z_j| = 1.$$

Since Y is the union of B_n, there exists $l\in\mathbb{Z}_+$ such that

$$\sum_{j=1}^{m} z_j u_{\alpha^j}\in B_l.$$

Choose $k\in\mathbb{Z}_+$, $k\geqslant l$ for which $\alpha^{j,k}=(\alpha^j_0,\dots\alpha^j_k)\in\mathbb{D}_k$ are pairwise different for $1\leqslant j\leqslant m$. From (S1) it follows that $u_{\alpha^j}\in\overline{U}_{\alpha^{j,k}}$ for $1\leqslant j\leqslant m$. According to (S2),

$$\sum_{j=1}^{m} z_j u_{\alpha^j}\notin B_k\supseteq B_l.$$

The last two displays contradict each other. Thus, $\{u_\alpha\}_{\alpha\in\{0,1\}^{\mathbb{Z}_+}}$ is linearly independent modulo Y. □

LEMMA **4.6.** *Let X be e Fréchet space and Y be a linear subspace of X carrying a stronger Fréchet space topology and $\mu=\dim_{\mathbb{C}}X/Y$. If μ is finite then Y is closed in X, if μ is infinite then $\mu\geqslant\mathfrak{c}$.*

Proof. First, suppose that μ is finite. Then there exists a finite dimensional linear subspace E of X such that $E\oplus Y=X$ in the algebraic sense. Consider the linear map T from the Fréchet space $Y\times E$ to the Fréchet space X given by $T(u,y)=u+y$. Since the topology of Y is stronger than the one inherited from X, we see that T is continuous. Since $E\oplus Y=X$, we see that T is invertible. According to the Banach inverse mapping theorem [10], T^{-1} is continuous and therefore $Y=(T^{-1})^{-1}(Y\times\{0\})$ is closed as a pre-image of the closed set $Y\times\{0\}$ with respect to the continuous map T^{-1}.

Assume now that μ is infinite. Choose a base $\{U_n\}_{n\in\mathbb{Z}_+}$ of convex symmetric neighborhoods of zero in Y. Symbols $\overline{U}_n$ stand for the closures of U_n in X. For any $n\in\mathbb{Z}_+$ let Y_n be the linear span of $\overline{U}_n$. Since $Y_n=\bigcup\limits_{k=1}^{\infty}k\overline{U}_n$, each Y_n is the union of countably many closed subsets of X. Denote also $Z=\bigcap\limits_{n=0}^{\infty}Y_n$. Note that if $\{V_k\}_{k\in\mathbb{Z}_+}$ is a base of neighborhoods of zero in X, then $\{V_k\cap\overline{U}_n\}_{k\in\mathbb{Z}_+}$ is a base of neighborhoods[1] of zero of a Fréchet space topology on Y_n, stronger than the one inherited from X.

Case 1: there exists $n\in\mathbb{Z}_+$ for which the codimension of Y_n in X is infinite. According to Lemma 4.5 $\dim_{\mathbb{C}}X/Y_n\geqslant\mathfrak{c}$. Since $Y\subseteq Y_n$, we have $\mu\geqslant\mathfrak{c}$ as required.

Case 2: the codimension of Z in X is infinite and for any $n\in\mathbb{Z}_+$ the codimension of Y_n in X is finite. According to the already proven first part of the lemma, any Y_n is closed in X and therefore Z is closed in X. Hence the infinite dimensional Fréchet space X/Z has algebraic dimension $\geqslant\mathfrak{c}$ according to Corollary 4.3. Since $Y\subseteq Z$, we have $\mu\geqslant\mathfrak{c}$ as required.

Case 3: the codimension of Z in X is finite. As in the previous case Z is closed in X as the intersection of closed linear subspaces Y_n. Finiteness of the codimension of Z implies that $Y_n=Z$ for sufficiently large n. Consider the identity embedding $J:Y\to Z$, where Z carries the Fréchet space topology inherited from X. Since $Y_n=Z$ for sufficiently large n, it follows that J is almost open [3]. Since any almost open continuous linear map between Fréchet spaces is surjective and open [3], we see that J is onto. Thus, $Y=Z$ and therefore Y has finite codimension in X. This contradiction shows that Case 3 does not occur. □

4.2 Density of ranges

For a family $\{\tau_\alpha\}_{\alpha\in A}$ of topologies on a set X, the symbol $\bigvee\limits_{\alpha\in A}\tau_\alpha$ stands for the topology, whose base is formed by the sets $\bigcap\limits_{j=1}^{n}U_j$, where $U_j\in\tau_{\alpha_j}$ and $\alpha_j\in A$. In other words, τ is the weakest topology stronger than each τ_α.

LEMMA **4.7.** *Let (X_n,τ_n) for $n\in\mathbb{Z}_+$ be Fréchet spaces such that X_{n+1} is a linear subspace of X_n and $\tau_n\big|_{X_{n+1}}\subseteq\tau_{n+1}$ for each $n\in\mathbb{Z}_+$. Let also $Y=\bigcap\limits_{n=0}^{\infty}X_n$ be endowed with the topology $\tau=\bigvee\limits_{n=0}^{\infty}\tau_n\big|_Y$. Then (Y,τ) is a Fréchet space. Moreover, if X_{n+1} is τ_n-dense in X_n for each $n\in\mathbb{Z}_+$,*

[1]We do not assume neighborhoods to be open. A neighborhood of a point x of a topological space X is a set containing an open set containing x.

then Y is τ_n-dense in X_n for each $n \in \mathbb{Z}_+$.

Proof. The topology τ is metrizable since each τ_n is metrizable. Let $\{x_k\}_{k\in\mathbb{Z}_+}$ be a Cauchy sequence in (Y,τ). Since τ is stronger than the restriction to Y of any τ_n, $\{x_k\}_{k\in\mathbb{Z}_+}$ is a Cauchy sequence in (X_n,τ_n) for any $n\in\mathbb{Z}_+$. Hence for each $n\in\mathbb{Z}_+$, x_k converges to $u_n\in X_n$ with respect to τ_n. Since the restriction of τ_n to X_{n+1} is weaker than τ_{n+1}, x_n is τ_n convergent to u_{n+1} in X_n. The uniqueness of a limit of a sequence in a Hausdorff topological space implies that $u_{n+1}=u_n$ for each $n\in\mathbb{Z}_+$. Hence there exists $u\in Y$ such that $u_n=u$ for each $n\in\mathbb{Z}_+$. Since x_k is τ_n-convergent to u for any $n\in\mathbb{Z}_+$, we see that x_k is τ-convergent to u. The completeness of (Y,τ) is proved.

The density part follows from the Mittag–Leffler lemma. It is also a particular case of Lemma 3.2.2 [16], dealing with projective limits of sequences of complete metric spaces. □

Proposition **4.8.** *Let T be an injective continuous linear operator with dense range acting on a Fréchet space X. Then X_T is dense in X. Moreover, X_T carries a Fréchet space topology stronger than the topology inherited from X, with respect to which the restriction T_0 of T to X_T is continuous.*

Proof. Let $\{p_k\}_{k\in\mathbb{Z}_+}$ be a sequence of seminorms defining the initial topology τ_0 on $X_0=X$. Consider the topology τ_n on $X_n=T^n(X)$ given by the sequence of seminorms

$$p_{n,k}(x)=\sum_{j=0}^{n}p_k(T^{-j}(x)),\quad k\in\mathbb{Z}_+.$$

One can easily verify that (X_n,τ_n) is a Fréchet space and $\tau_n\big|_{X_{n+1}}\subseteq\tau_{n+1}$ for each $n\in\mathbb{Z}_+$. Moreover, density of $T(X)$ in X implies τ_n-density of $T(X_n)=X_{n+1}$ in X_n for each $n\in\mathbb{Z}_+$. Indeed, the restriction of T to X_n considered as a linear operator on the Fréchet space (X_n,τ_n) is similar to T with the continuous invertible operator $G=T^n:X\to X_n$, providing the similarity. By Lemma 4.7 $X_T=\bigcap\limits_{n=0}^{\infty}X_n$ is dense in X and X_T with the topology $\tau=\bigvee\limits_{0}^{\infty}\tau_n$ is a Fréchet space. One can easily verify that T_0 is τ-continuous. □

4.3 Weak boundedness

Definition **3.** A sequence $\{f_n\}_{n\in\mathbb{Z}_+}$ of linear functionals on a linear space X is said to be *weakly bounded* if there exists a sequence $\{b_n\}_{n\in\mathbb{Z}_+}$ of positive numbers such that for any $x\in X$ there is $c(x)>0$ for which $|f_n(x)|\leqslant c(x)b_n$ for each $n\in\mathbb{Z}_+$.

Lemma **4.9.** *Let $\{f_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous linear functionals on a Fréchet space X. Then $\{f_n\}_{n\in\mathbb{Z}_+}$ is weakly bounded if and only if there exists a continuous seminorm p on X, with respect to which all f_n are bounded.*

Proof. Suppose that each f_n is bounded with respect to a continuous seminorm p on X. Then

$$b_n=\sup_{p(x)\leqslant 1}|f_n(x)|<\infty\ \text{ for any }\ n\in\mathbb{Z}_+.$$

Clearly $|f_n(x)|\leqslant p(x)b_n$ for any $x\in X$ and any $n\in\mathbb{Z}_+$, which means that $\{f_n\}_{n\in\mathbb{Z}_+}$ is weakly bounded.

Suppose now that $\{f_n\}_{n\in\mathbb{Z}_+}$ is weakly bounded and $\{b_n\}_{n\in\mathbb{Z}_+}$ is a sequence of positive integers such that $f_n(x)=O(b_n)$ as $n\to\infty$ for each $x\in X$. Then

$$p(x)=\sup_{n\in\mathbb{Z}_+}\frac{|f_n(x)|}{b_n}$$

is a seminorm on X, with respect to which all f_n are bounded. It remains to verify that p is continuous. Clearly the unit p-ball $W_p = \{x \in X : p(x) \leqslant 1\}$ satisfies $W_p = \bigcap_{n=0}^{\infty} U_n$, where $U_n = \{x \in X : |f_n(x)| \leqslant b_n\}$. Since f_n are continuous, we see that the sets U_n are closed. Therefore W_p is closed. Thus, W_p is a barrel, that is a closed convex balanced absorbing subset of X. Since any Fréchet space is barreled [3], W_p is a neighborhood of zero in X. Hence p is continuous. □

5 Continuous tame operators

We start with a criterion of tameness for general linear operators.

LEMMA **5.1.** *Let T be a linear operator on a linear space X. Then T is tame if and only if for any sequence $\{y_n\}_{n\in\mathbb{Z}_+}$ of elements of $X \setminus T(X)$ and any sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers there exists $x \in X$ such that*

$$x \equiv \sum_{j=0}^{n} c_j T^j y_j \ (\mathrm{mod} T^{n+1}(X)) \tag{24}$$

for each $n \in \mathbb{Z}_+$.

Proof. Clearly (4) with $x_j = c_j y_j$ is exactly (24) and the 'if' part of Lemma 3.4 follows. It remains to prove the 'only if' part. Suppose that for any sequence $\{y_n\}_{n\in\mathbb{Z}_+}$ of elements of $X \setminus T(X)$ and any sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers there exists $x \in X$ such that (24) is satisfied for each $n \in \mathbb{Z}_+$. We have to prove that T is tame. If $X = T(X)$ the result is trivial. Suppose that $X \neq T(X)$ and fix $u \in X \setminus T(X)$. Let $\{x_n\}_{n\in\mathbb{Z}_+}$ be a sequence of elements of X. It suffices to construct sequences $\{y_n\}_{n\in\mathbb{Z}_+}$ of elements of $X \setminus T(X)$ and $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers such that for any $n \in \mathbb{Z}_+$ validity of (4) is equivalent to validity of (24).

On step 0 we put $y_0 = x_0$, $c_0 = 1$ if $x_0 \notin T(X)$ and $y_0 = u$, $c_0 = 0$ if $x_0 \in T(X)$. Obviously (4) is equivalent to (24) for $n = 0$. Suppose now that m is a positive integer and $y_0, \dots, y_{m-1} \in X \setminus T(X)$ and $c_0, \dots, c_{m-1} \in \mathbb{C}$ are such that (4) is equivalent to (24) for $0 \leqslant n \leqslant m-1$. To say that (4) is equivalent to (24) is the same as to say that

$$\sum_{j=0}^{n} T^j(x_j - c_j y_j) \in T^{n+1}(X).$$

Since (4) is equivalent to (24) for $n = m-1$, there exists $w \in X$ such that

$$\sum_{j=0}^{m-1} T^j(x_j - c_j y_j) = T^m w.$$

If $w + x_m \in T(X)$, we put $y_m = u$ and $c_m = 0$. If $w + x_m \notin T(X)$, we put $y_m = w + x_m$ and $c_m = 1$. In any case we have

$$\sum_{j=0}^{m} T^j(x_j - c_j y_j) = T^m(u + x_m - c_m y_m) \in T^{m+1}(X).$$

Thus, (4) is equivalent to (24) for $n = m$. The inductive procedure is complete and so is the proof of the lemma. □

LEMMA **5.2.** *Let T be a linear operator acting on a linear space X, $n \in \mathbb{Z}_+$, Y be a T-invariant linear subspace of X such that $T^n(X) \subseteq Y$ and $S : Y \to Y$ be the restriction of T to Y. Then tameness of S implies tameness of T.*

Proof. Suppose that S is tame. For $n = 0$ we have $S = T$ and the result is trivial. Thus, we can assume that $n > 0$. Let $\{x_k\}_{k\in\mathbb{Z}_+}$ be a sequence of elements of X. Then $\{T^n x_{n+k}\}_{k\in\mathbb{Z}_+}$ is a sequence of elements of $T^n(X) \subseteq Y$. Since S is tame, there exists $y \in Y$ such that

$$y - \sum_{j=0}^{m} T^{n+j} x_{n+j} \in S^{m+1}(Y) \subseteq T^{m+1}(X) \ \text{ for each } m \in \mathbb{Z}_+.$$

Hence,

$$y - \sum_{l=n}^{m} T^l x_l \in T^{m+1}(X) \ \text{ for } m \geqslant n.$$

Let $x = y + \sum\limits_{j=0}^{n-1} T^j x_j$. From the last display it follows that

$$x - \sum_{j=0}^{m} T^j x_j \in T^{m+1}(X) \ \text{ for any } m \in \mathbb{Z}_+.$$

Hence T is tame. □

COROLLARY...

LEMMA **5.3.** *Let T be an injective linear operator on a linear space X, $x \in X$ and $y_0, \dots, y_n \in X \setminus T(X)$ be such that* (24) *is satisfied for some $c_0, \dots, c_n \in \mathbb{C}$. Then the numbers $c_0, \dots, c_n$ are uniquely determined by $x, y_0, \dots, y_n$.*

Proof. Suppose that (24) is also satisfied with c_j replaced by $c'_j \in \mathbb{C}$. We have to prove that $c_j = c'_j$ for $0 \leqslant j \leqslant n$. Since $\sum\limits_{j=0}^{n} (c_j - c'_j) T^j y_j \in T^{n+1}(X)$, we see that $(c_0 - c'_0) y_0 \in T(X)$. Then $c_0 = c'_0$ because $y_0 \notin T(X)$. Hence $\sum\limits_{j=1}^{n} (c_j - c'_j) T^j y_j \in T^{n+1}(X)$ and therefore $(c_1 - c'_1) T y_1 \in T^2(X)$. Since T is injective, $(c_1 - c'_1) y_1 \in T(X)$. Then $c_1 = c'_1$ since $y_1 \notin T(X)$. Proceeding in the same way, we obtain that $c_j = c'_j$ for each $j \leqslant n$. □

We shall introduce some additional notation. Let T be an injective linear operator on a linear space X and $\varkappa = \{y_n\}_{n\in\mathbb{Z}_+}$ be a sequence of elements of $X \setminus T(X)$. Symbol $X_{T}(\varkappa)$ stands for the set of $x \in X$ such that for any positive integer $n \in \mathbb{Z}_+$ there exist $c_0, \dots, c_n \in \mathbb{C}$ for which (24) is satisfied. According to Lemma 5.3 the numbers c_j depend only on x. Thus, the maps $x \mapsto c_j$ are well-defined linear functionals on the linear space $X_{T}(\varkappa)$. We denote them by symbols $\Phi_j = \Phi_j^{T,\varkappa}$. In this new notation (24) can be rewritten as

$$x \equiv \sum_{j=0}^{n} \Phi_j(x) T^j y_j \ (\mathrm{mod}\, T^{n+1}(X)), \tag{25}$$

which holds true for each $x \in X_{T}(\varkappa)$ and each $n \in \mathbb{Z}_+$. Then Lemma 5.1 immediately implies the following corollary.

COROLLARY **5.4.** *Let T be an injective linear operator on a linear space X. Then T is tame if and only if for any sequence $\varkappa = \{y_n\}_{n\in\mathbb{Z}_+}$ of elements of $X \setminus T(X)$ and any sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers, there exists $x \in X_{T}(\varkappa)$ such that $\Phi_n^{T,\varkappa}(x) = c_n$ for each $n \in \mathbb{Z}_+$.*

5.1 Tameness of continuous linear operators on Fréchet spaces

In order to show that certain continuous linear operators on Fréchet spaces are tame we need an old result of Eidelheit [4], related to the abstract moment problem. We present it in a slightly different form obviously equivalent to the original one. A different proof can be found in [12].

PROPOSITION **5.5.** *Let X be a Fréchet space and $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ be a sequence of continuous linear functionals on X. Then the following conditions are equivalent:*

(E1) *for any sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers there exists $x\in X$ such that $\varphi_n(x)=c_n$ for each $n\in\mathbb{Z}_+$;*

(E2) *the family $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ of functionals is linearly independent and for any continuous seminorm p on X the space*

$$E_p=\{\varphi\in\ \mathrm{span}\,\{\varphi_j:j\in\mathbb{Z}_+\}:\varphi \text{ is } p\text{-bounded}\}$$

is finite dimensional.

LEMMA **5.6.** *Let X be a Fréchet space and $T:X\to X$ be an injective continuous linear operator such that X_{T} is dense in X. Then T is tame.*

Proof. Let $\varkappa=\{y_n\}_{n\in\mathbb{Z}_+}$ be a sequence of elements of $X\setminus T(X)$. Since $y_j\notin T(X)$, the vectors $y_0,Ty_1,\dots,T^ny_n$ are linearly independent modulo $T^{n+1}(X)$ for any $n\in\mathbb{Z}_+$. Therefore the space

$$E^{\varkappa}_n=\ \mathrm{span}\,\{T^jy_j:0\leqslant j\leqslant n\}$$

is $n+1$-dimensional and $E^{\varkappa}_n\cap T^{n+1}(X)=\{0\}$. Denote $X^{\varkappa}_n=E^{\varkappa}_n\oplus T^{n+1}(X)$. Clearly

$$X_{T}(\varkappa)=\bigcap_{n=0}^{\infty}X^{\varkappa}_n.$$

Let $\{p_k\}_{k\in\mathbb{Z}_+}$ be a sequence of seminorms defining the topology of X. Since T is injective, any $x\in X^{\varkappa}_n$ can be uniquely written in the form

$$x=\sum_{j=0}^{n}c_jT^jx_j+T^{n+1}u,\quad \text{where}\quad c_j\in\mathbb{C},\ u\in X.$$

This allows us to define seminorms

$$p_{n,k}(x)=\sum_{j=0}^{n}|c_j|+p_k(u)$$

on $X^{\varkappa}_n$. The sequence $\{p_{n,k}\}_{k\in\mathbb{Z}_+}$ of seminorms on $X^{\varkappa}_n$ defines a metrizable locally convex topology τ_n on $X^{\varkappa}_n$. Since T is continuous, it follows that the restriction of τ_n to $X^{\varkappa}_{n+1}$ is weaker than τ_{n+1}. Moreover, the map $x\mapsto(c_0,\dots,c_n,u)$ is an isomorphism between $(X^{\varkappa}_n,\tau_n)$ and the Fréchet space $\mathbb{C}^{n+1}\times X$. Hence $(X^{\varkappa}_n,\tau_n)$ is a Fréchet space for each $n\in\mathbb{Z}_+$. By Lemma 4.7 $X_{T}(\varkappa)$ endowed with the topology $\tau=\bigvee\limits_{n=0}^{\infty}\tau_n$ is a Fréchet space. Clearly τ is defined by the family of seminorms $\{p_{n,k}\}_{n,k\in\mathbb{Z}_+}$. Since for any $n\in\mathbb{Z}_+$, the functional $\Phi_n=\Phi^{T,\varkappa}_n$ is bounded with respect to $p_{n,0}$, we see that all Φ_n are τ-continuous linear functionals on $X_{T}(\varkappa)$. According to Corollary 5.4 it suffices to verify that for any sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ of complex numbers, there exists $x\in X_{T}(\varkappa)$ for

which $\Phi_n(x) = c_n$ for each $n \in \mathbb{Z}_+$. Since $\{(T^n y_n, \Phi_n)\}_{n\in\mathbb{Z}_+}$ is a biorthogonal sequence, Φ_n are linearly independent. Let $E = \text{span}\,\{\Phi_n : n \in \mathbb{Z}_+\}$. By Proposition 5.5 it suffices to show that for any τ-continuous seminorm p on $X_T(\varkappa)$,

$$\text{the space } \ E_p = \{\varphi \in E : \varphi \text{ is } p\text{-bounded}\} \ \text{ is finite dimensional.}$$

Let p be a τ-continuous seminorm on $X_T(\varkappa)$. Since $\tau = \bigvee\limits_{k=0}^{\infty} \tau_k$ and restriction of τ_k to $X^{\varkappa}_{k+1}$ is weaker than τ_{k+1} for each $k \in \mathbb{Z}_+$, we see that there exists $n \in \mathbb{Z}_+$ for which p is τ_n-continuous.

Since T^{n+1} acting from X to the subspace $T^{n+1}(X)$ of $X^{\varkappa}_n$, endowed with the topology τ_n, is an isomorphism of Fréchet spaces, mapping X_T onto itself, we have that X_T is τ_n-dense in $T^{n+1}(X)$. Since any $\varphi \in E$ vanishes on X_T, we see that any τ_n-continuous $\varphi \in E$ vanishes on $T^{n+1}(X) \cap X_T(\varkappa)$. Therefore

$$E_p \subseteq \{\varphi \in E : \varphi \text{ is } \tau_n\text{-continuous}\} \subseteq \{\varphi \in E : \ \varphi \text{ vanishes on } T^{n+1}(X) \cap X_T(\varkappa)\}.$$

The dimension of the last space does not exceed the codimension of $T^{n+1}(X) \cap X_T(\varkappa)$ in $X_T(\varkappa)$, which does not exceed the codimension of $T^{n+1}(X)$ in $X^{\varkappa}_n$, which is finite. □

Proposition **5.7.** *Let X be a Fréchet space and $T : X \to X$ be an injective continuous linear operator such that there exists $m \in \mathbb{Z}_+$ for which $\overline{T^m(X)} = \overline{T^{m+1}(X)}$. Then T is tame.*

Proof. Clearly $Y = \overline{T^m(X)}$ is a closed T-invariant subspace of X. Let $S : Y \to Y$ be the restriction of T to Y. Condition $\overline{T^m(X)} = \overline{T^{m+1}(X)}$ implies that the range of S is dense. By Proposition 4.8 Y_S is dense in Y. Applying Lemma 5.6, we see that S is tame. From Lemma 5.2 it follows that T is tame. □

5.2 Bounded tame operators on Banach spaces

In the Banach space setting the sufficient condition of tameness in Proposition 5.7 turns out to be also necessary.

Theorem **5.8.** *Let T be an injective bounded linear operator on a Banach space X. Then T is tame if and only if there exists $m \in \mathbb{Z}_+$ for which $\overline{T^m(X)} = \overline{T^{m+1}(X)}$.*

Proof. By Proposition 5.7 the existence of $m \in \mathbb{Z}_+$ for which $\overline{T^m(X)} = \overline{T^{m+1}(X)}$ implies tameness of T. Suppose now that $\overline{T^m(X)} \neq \overline{T^{m+1}(X)}$ for any $m \in \mathbb{Z}_+$. Then we can choose a sequence $\varkappa = \{y_n\}_{n\in\mathbb{Z}_+}$ in X such that $T^n y_n \notin \overline{T^{n+1}(X)}$ for each $n \in \mathbb{Z}_+$. In particular, $\varkappa$ is a sequence of elements of $X \setminus T(X)$. Using the Hahn–Banach theorem, for any $n \in \mathbb{Z}_+$, we can find a continuous linear functional φ_n on X such that $\varphi_n(T^n y_n) = 1$, $\varphi_n(T^j y_j) = 0$ for $j < n$ and $\varphi_n(x) = 0$ for each $x \in \overline{T^{n+1}(X)}$. From the definition of the functionals $\Phi_n = \Phi^{T,\varkappa}_n$ on the Fréchet space $X_T(\varkappa)$ it follows that each Φ_n is the restriction of φ_n to $X_T(\varkappa)$. Hence,

$$|\Phi_n(x)| = |\varphi_n(x)| \leqslant \|\varphi_n\| \cdot \|x\|_X \ \text{ for any } x \in X_T(\varkappa) \text{ and any } n \in \mathbb{Z}_+.$$

Therefore there exists no $x \in X_T(\varkappa)$ such that $\Phi_n(x) = n\|\varphi_n\|$ for each $n \in \mathbb{Z}_+$. According to Corollary 5.4, T is not tame. □

6 Similarity to the Volterra operator

Let $\mathcal{V}$ be the class of injective quasinilpotent continuous linear operators T acting on a Fréchet space X of algebraic dimension $\mathfrak{c}$ such that the codimension of $T(X)$ in X is infinite and there exists $m \in \mathbb{Z}_+$ for which $\overline{T^{m+1}(X)} = \overline{T^m(X)}$.

Proposition **6.1.** *Any $T \in \mathcal{V}$ is similar to the Volterra operator V acting on the Banach space $\mathcal{C} = C[0,1]$.*

Proof. Let $T \in \mathcal{V}$. Since the codimension of $T(X)$ in X is infinite and $T(X)$ carries a stronger Fréchet topology (namely, the one transferred from X by the operator T), from Lemma 4.6 it follows that $\dim_{\mathbb{C}} X/T(X) \geqslant \mathfrak{c}$. Since $\dim_{\mathbb{C}} X = \mathfrak{c}$, we have $\dim_{\mathbb{C}} X/T(X) = \mathfrak{c}$. Condition $\overline{T^{m+1}(X)} = \overline{T^m(X)}$ together with Proposition 4.8 imply that X_T is dense in $\overline{T^m(X)}$ and therefore X_T is non-zero. Moreover, X_T carries a stronger Fréchet space topology, with respect to which the restriction T_0 of T to X_T is continuous. According to Lemma 3.1 $\sigma(T_0) = \varnothing$. By Proposition 4.4 and Corollary 4.3, $\dim_{\mathcal{R}} X_T \geqslant \mathfrak{c}$. Since $\dim_{\mathbb{C}} X = \mathfrak{c}$, we have $\dim_{\mathcal{R}} X_T = \mathfrak{c}$. Using Proposition 5.7, we see that T is tame.

Now from Theorem 3.2 it follows that any two operators from $\mathcal{V}$ are similar. Since V has non-closed range, Lemma 4.6 implies that $V(\mathcal{C})$ has infinite codimension in $\mathcal{C}$. Clearly V is injective, quasinilpotent and $\overline{V(\mathcal{C})} = \overline{V^2(\mathcal{C})}$. Hence $V \in \mathcal{V}$. □

6.1 Proof of Theorem 1.2

Since the range of any injective quasinilpotent operator on a Banach space is non-closed, Lemma 4.6 implies that $\dim_{\mathbb{C}} X/T(X) \geqslant \mathfrak{c}$. Using Proposition 6.1, we obtain that (C2) implies (C1). Suppose that (C1) is satisfied. Since by Theorem 5.8 V acting on $C[0,1]$ is tame and tameness is a similarity invariant, we observe that T is tame. Since quasinilpotence and injectivity are also preserved by similarity, we see that T is injective and quasinilpotent. Since T is tame we, using Theorem 5.8 once again, obtain that there exists $m \in \mathbb{Z}_+$ for which $\overline{T^{m+1}(X)} = \overline{T^m(X)}$. Thus, (C2) is satisfied. The proof of Theorem 1.2 is complete.

7 Proof of Theorem 1.5

We start with the following general fact.

Lemma **7.1.** *Let $\{H_n\}_{n\in\mathbb{Z}_+}$ be a strictly decreasing sequence of closed finite codimensional linear subspaces of a Fréchet space X such that $\bigcap\limits_{n=0}^{\infty} H_n = \{0\}$. Let also Y be a Fréchet space and $G : X \to Y$ be an invertible linear operator such that $G(H_n)$ is closed in Y for any $n \in \mathbb{Z}_+$. Then G is continuous.*

Proof. Choose a decreasing sequence $\{K_m\}_{m\in\mathbb{Z}_+}$ of closed linear subspaces of X such that $K_0 = X$, $\dim_{\mathbb{C}} K_m/K_{m+1} = 1$ for each $n \in \mathbb{Z}_+$ and $\{H_n\}_{n\in\mathbb{Z}_+}$ is a subsequence of $\{K_m\}_{m\in\mathbb{Z}_+}$: $H_n = K_{m_n}$ for some strictly increasing sequence of non-negative integers $\{m_n\}_{n\in\mathbb{Z}_+}$. Pick $x_n \in K_n \backslash K_{n+1}$. Using the Hahn–Banach theorem, we can choose continuous linear functionals $\varphi_n : X \to \mathbb{C}$ such that $\varphi_n(x_n) = 1$, $\varphi_n(x_j) = 0$ for $j < n$ and $\varphi_n\big|_{K_{n+1}} \equiv 0$. Consider the functionals $\psi_n : Y \to \mathbb{C}$ defined by the formulas $\psi_n(y) = \varphi_n(G^{-1}y)$. Each ψ_n vanishes on a finite codimensional closed linear space of the shape $G(H_m)$ and therefore is continuous. Since

$$\bigcap_{m=0}^{\infty} \ker \varphi_m = \bigcap_{m=0}^{\infty} K_m = \bigcap_{n=0}^{\infty} H_n = \{0\},$$

we see that the functionals $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ separate points of X. Since G is invertible, it follows that $\{\psi_n\}_{n\in\mathbb{Z}_+}$ separate points of Y. Consider the topologies σ on X and σ^* on Y defined by the families of functionals $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ and $\{\psi_n\}_{n\in\mathbb{Z}_+}$ respectively. Recall that this means that σ and σ^*

are given by the seminorms

$$p_k(x) = \sum_{j=0}^{k} |\varphi_j(x)| \ \text{ and } \ q_k(y) = \sum_{j=0}^{k} |\psi_j(y)| \quad (k \in \mathbb{Z}_+)$$

respectively. Continuity of φ_n and ψ_n implies that σ and σ^* are weaker than the initial topologies on X and Y. Since φ_n separate the points of X and ψ_n separate the points of Y, the topologies σ and σ^* are Hausdorff. From the formula $\psi_n(y) = \varphi_n(G^{-1}y)$ it immediately follows that G is σ-σ^* continuous. Therefore the graph Γ_G is closed in the product of (X, σ) and (Y, σ^*) as is the graph of any continuous map between Hausdorff topological spaces. Since σ and σ^* are weaker than the initial topologies on X and Y, we see that Γ_G is closed in $X \times Y$. The Banach Closed Graph Theorem [10, 3] implies that G is continuous. $\Box$

We are going to use notation from Section 5. For an injective linear operator T on a linear space X symbol $B(T, X)$ stands for the set of vectors $y \in X$ such that either $y \in T(X)$ or $y \notin T(X)$ and there exists a sequence $\{u_n\}_{n\in\mathbb{Z}_+}$ of elements of X for which the sequence $\{\Phi_n^{T,\varkappa}\}_{n\in\mathbb{Z}_+}$ of linear functionals on the space $X_{T}(\varkappa)$ is not weakly bounded, where $\varkappa = \{y + Tu_n\}_{n\in\mathbb{Z}_+}$.

For a bounded linear operator T on a Banach space X we denote

$$A(T, X) = \bigcup_{n=0}^{\infty} T^{-n}(\overline{T^{n+1}(X)}).$$

Clearly $A(T, X)$ is a linear subspace of X as a union of the increasing sequence of linear subspaces $T^{-n}(\overline{T^{n+1}(X)})$.

Lemma **7.2.** *Let T be an injective bounded linear operator on a Banach space X. Then $B(T, X) \subseteq A(T, X)$.*

Proof. Let $y \in X \setminus A(T, X)$. Then $T^n y \notin \overline{T^{n+1}(X)}$ for each $n \in \mathbb{Z}_+$. Let also $\{u_n\}_{n\in\mathbb{Z}_+}$ be a sequence of elements of X and $y_n = y + Tu_n$. Then $T^n y_n \notin \overline{T^{n+1}(X)}$ for each $n \in \mathbb{Z}_+$. Using the Hahn–Banach theorem, we can choose continuous linear functionals φ_n on X such that $\varphi_n(T^n y_n) = 1$, $\varphi_n(T^j y_j) = 0$ for $j < n$ and $\varphi_n\big|_{\overline{T^{n+1}(X)}} \equiv 0$. Then the functionals $\Phi_n = \Phi_n^{T,\varkappa}$ on $X_{T}(\varkappa)$ for $\varkappa = \{y_n\}_{n\in\mathbb{Z}_+}$ coincide with the restrictions of φ_n to $X_{T}(\varkappa)$. Hence $|\Phi_n(x)| = |\varphi_n(x)| \leqslant \|\varphi_n\|\|x\|$ for any $x \in X_{T}(\varkappa)$ and any $n \in \mathbb{Z}_+$ and therefore the sequence $\{\Phi_n\}_{n\in\mathbb{Z}_+}$ is weakly bounded. Thus, $y \notin B(T, X)$, which proves the desired inclusion. $\Box$

Lemma **7.3.** *Let T be an injective bounded linear operator on a Banach space X and $\varkappa = \{y_n\}_{n\in\mathbb{Z}_+}$ be a sequence of elements of $X \setminus T(X)$ such that the sequence $\{\Phi_n = \Phi_n^{T,\varkappa}\}_{n\in\mathbb{Z}_+}$ of linear functionals on $X_{T}(\varkappa)$ is weakly bounded. Then there exists $k \in \mathbb{Z}_+$ such that for any $n \geqslant k$, $T^{n-k} y_n$ does not belong to the closure of $\operatorname{span}\{T^{n-k+j} y_{n+j} : j \geqslant 1\}$ in X.*

Proof. As in Section 5, $E_n^\varkappa = \operatorname{span}\{T^j y_j : 0 \leqslant j \leqslant n\}$ and $X_n^\varkappa = E_n^\varkappa \oplus T^{n+1}(X)$. We endow $X_n^\varkappa$ with the norm

$$p_n\left(T^{n+1}u + \sum_{j=0}^{n} c_j T^j y_j\right) = \|u\|_X + \sum_{j=0}^{n} |c_j|.$$

Then $(X_n^\varkappa, p_n)$ is a Banach space. Indeed, the map $(c, u) \mapsto T^{n+1}u + \sum\limits_{j=0}^{n} c_j T^j y_j$ from $\mathbb{C}^{n+1} \times X$ is an isomorphism of normed spaces and $\mathbb{C}^{n+1} \times X$ is complete. Continuity of T implies that there exist $c_n > 0$ for which

$$p_n(x) \leqslant c_n p_{n+1}(x) \ \text{ for any } n \in \mathbb{Z}_+ \text{ and any } x \in X_{n+1}^\varkappa. \tag{26}$$

According to Lemma 4.7 $X_T(\varkappa)=\bigcap\limits_{n=0}^{\infty} X_n^{\varkappa}$ endowed with the topology defined by the sequence $\{p_n\}_{n\in\mathbb{Z}_+}$ of norms is a Fréchet space. The functionals $\Phi_n=\Phi_n^{T,\varkappa}:X_T(\varkappa)\to\mathbb{C}$ are τ-continuous since each Φ_n is p_n-bounded. Since the sequence $\{\Phi_n\}_{n\in\mathbb{Z}_+}$ is weakly bounded, Lemma 4.9 implies the existence of a τ-continuous seminorm p on $X_T(\varkappa)$, with respect to which each Φ_n is bounded. According to (26), there is a positive integer k such that each Φ_n is p_{k-1}-bounded. Hence for any $n\in\mathbb{Z}_+$, $\ker\Phi_n$ is p_{k-1}-closed in $X_T(\varkappa)$. Since

$$T^ny_n\notin\ker\Phi_n\ \ \text{and}\ \ \mathrm{span}\,\{T^my_m:m\neq n+1\}\subset\ker\Phi_n,$$

we see that T^ny_n does not belong to the p_{k-1}-closure of $\mathrm{span}\,\{T^my_m:m\geqslant n+1\}$. From the definition of the norm p_{k-1} it follows that for $n\geqslant k$, the last condition means exactly that $T^{n-k}y_n$ does not belong to the closure in X of

$$\mathrm{span}\,\{T^{m-k}y_m:m\geqslant n+1\}=\mathrm{span}\,\{T^{n-k+j}y_{n+j}:j\geqslant 1\}.\ \ \Box$$

Let T be an injective bounded linear operator on a Banach space X. It is easy to see that for any $n\in\mathbb{Z}_+$, the operator

$$T_n:X/T^{-n}(\overline{T^{n+1}(X)})\to\overline{T^n(X)}/\overline{T^{n+1}(X)},\quad T_n(x+T^{-n}(\overline{T^{n+1}(X)}))=T^nx+\overline{T^{n+1}(X)}$$

is an injective bounded linear operator with dense range. It follows that if one of the spaces $X/T^{-n}(\overline{T^{n+1}(X)})$ or $\overline{T^n(X)}/\overline{T^{n+1}(X)}$ is finite dimensional then the other has the same dimension and T_n is an isomorphism between them. Moreover for each $n\in\mathbb{Z}_+$, the operator

$$\widetilde{T}_n:\overline{T^n(X)}/\overline{T^{n+1}(X)}\to\overline{T^{n+1}(X)}/\overline{T^{n+2}(X)},\quad \widetilde{T}_n(x+\overline{T^{n+1}(X)})=Tx+\overline{T^{n+2}(X)}$$

is a bounded linear operator with dense range. Thus, if $\overline{T^n(X)}/\overline{T^{n+1}(X)}$ is finite dimensional then so is $\overline{T^{n+1}(X)}/\overline{T^{n+2}(X)}$ and $\widetilde{T}_n$ is onto. In particular, the dimension of $\overline{T^{n+1}(X)}/\overline{T^{n+2}(X)}$ does not exceed the dimension of $\overline{T^n(X)}/\overline{T^{n+1}(X)}$ and these dimensions coincide if and only if $\widetilde{T}_n$ is an isomorphism. In this case $T^{-n}(\overline{T^{n+1}(X)})=T^{-n-1}(\overline{T^{n+2}(X)})$. These observations are summarized in the following lemma.

LEMMA **7.4.** *Let $m\in\mathbb{Z}_+$ and T be an injective bounded linear operator on a Banach space X. For $n\in\mathbb{Z}_+$ let $\delta_n=\dim_{\mathbb{C}}\overline{T^{n+1}(X)}/\overline{T^n(X)}$ and $\delta'_n=\dim_{\mathbb{C}}X/T^{-n}(\overline{T^{n+1}(X)})$. Suppose also $\min\{\delta_m,\delta'_m\}$ is finite. Then $\delta_n=\delta'_n$ and $\delta_{n+1}\leqslant\delta_n$ for any $n\geqslant m$. In particular, there exists $m_0\in\mathbb{Z}_+$ and $k_0\in\mathbb{Z}_+$ such that $\delta_n=k_0$ for each $n\geqslant m_0$. Moreover, k_0 is exactly the codimension of $A(T,X)$ in X and if $n\geqslant m_0$ and $y\in\overline{T^{n+1}(X)}\setminus\overline{T^n(X)}$, then $Ty\in\overline{T^{n+2}(X)}\setminus\overline{T^{n+1}(X)}$.*

LEMMA **7.5.** *Let T be an injective bounded linear operator on a Banach space X such that (2) and (3) are satisfied. Consider $H_n\subseteq X$ defined inductively by the formulas $H_0=X$, $H_{k+1}=A(T,H_k)=\bigcup\limits_{n=0}^{\infty}T^{-n}(\overline{T^{n+1}(H_k)})$ for $k\in\mathbb{Z}_+$. Then $\{H_n\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of finite codimensional closed linear subspaces of X and $\bigcap\limits_{k=0}^{\infty}H_k=\{0\}$.*

P r o o f. Clearly H_1 is a linear subspace of X as a union of an increasing sequence $T^{-n}(\overline{T^{n+1}(X)})$ of linear subspaces. From (3) and Lemma 7.4 the sequence $T^{-n}(\overline{T^{n+1}(X)})$ stabilizes and $T^{-n}(\overline{T^{n+1}(X)})$ has finite codimension for sufficiently large n. Thus, $A(T,X)=T^{-n}(\overline{T^{n+1}(X)})$ for sufficiently large n and $H_1=A(T,X)$ is a closed finite codimensional subspace of X. Applying the same argument consecutively to the restrictions[2] of T to H_k, $k=1,2,\dots$ we see that H_{k+1} is closed and

[2]We use the obvious observation that if T satisfies (2) and (3) then so does any restriction of T to any invariant closed finite codimensional subspace.

finite codimensional in H_k for each $k \in \mathbb{Z}_+$. Thus, $\{H_n\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of finite codimensional closed linear subspaces of X. It remains to show that $\bigcap\limits_{k=0}^{\infty} H_k = \{0\}$.

Using Lemma 7.4, we see that for any fixed $k \in \mathbb{Z}_+$ there exists $m_k \in \mathbb{Z}_+$ such that $T^{-n}(\overline{T^{n+1}(H_k)})$ does not depend on n provided $n \geqslant m_k$. Thus, $H_{k+1} = T^{-n}(\overline{T^{n+1}(H_k)})$ for $n \geqslant m_k$. From this equality and the definition of H_k it follows that for any $k \in \mathbb{Z}_+$,

$$H_k \subseteq T^{-n}(\overline{T^{n+k}(X)}) \text{ for sufficiently large } n.$$

Let $y \in X$, $y \neq 0$. Using (2) we see that there exists $m \geqslant m_0$ such that $T^{m_0}y \in \overline{T^m(X)} \setminus \overline{T^{m+1}(X)}$. From the last statement in Lemma 7.4 it follows that $T^{m_0+l}y \in \overline{T^{m+l}(X)} \setminus \overline{T^{m+l+1}(X)}$ for each $l \in \mathbb{Z}_+$. In particular, $y \notin T^{-m_0-l}(\overline{T^{m+l+1}(X)})$ for each $l \in \mathbb{Z}_+$. According to the last display this means that $y \notin H_k$ for $k > m - m_0$. Thus, $\bigcap\limits_{k=0}^{\infty} H_k = \{0\}$. □

LEMMA **7.6.** *Let $m \in \mathbb{Z}_+$, T be an injective bounded linear operator on a Banach space X such that the codimension of $\overline{T^{m+1}(X)}$ in $\overline{T^m(X)}$ is finite, $y \in T^{-m}(\overline{T^{m+1}(X)})$ and A be an infinite subset of $\mathbb{Z}_+$ such that $0 \in A$. Then there exists a sequence $\{u_n\}_{n\in A}$ of elements of X such that $T^m(y + Tu_0)$ belongs to the closed linear span of $\{T^{m+n}(y + Tu_n) : n \in A \setminus \{0\}\}$.*

P r o o f. First, let us show that

$$T^n(X) + \overline{T^k(X)} = \overline{T^n(X)} \text{ for any } k, n \in \mathbb{Z}_+,\ k \geqslant n \geqslant m. \tag{27}$$

The inclusion $T^n(X) + \overline{T^k(X)} \subseteq \overline{T^n(X)}$ is obvious. According to Lemma 7.4, $\overline{T^k(X)}$ has finite codimension in $\overline{T^n(X)}$. Therefore $T^n(X) + \overline{T^k(X)}$ is a linear subspace of $\overline{T^n(X)}$, containing a closed finite codimensional subspace. Hence $T^n(X) + \overline{T^k(X)}$ is closed in $\overline{T^n(X)}$. On the other hand $T^n(X) + \overline{T^k(X)}$ contains $T^n(X)$ and therefore is dense in $\overline{T^n(X)}$. Thus, (27) is satisfied.

Let $\{j_n\}_{n\in\mathbb{Z}_+}$ be a strictly increasing sequence of non-negative integers such that $A = \{j_n : n \in \mathbb{Z}_+\}$. Clearly $j_0 = 0$.

We shall construct inductively a sequence $\{u_{j_n}\}_{n\in\mathbb{Z}_+}$ of elements of X such that for any $n \in \mathbb{Z}_+$,

$$\sum_{k=0}^{n} T^{j_k+m}(y + Tu_{j_k}) \in \overline{T^{m+j_{n+1}+1}(X)} \text{ and } \left\| \sum_{k=0}^{n} T^{j_k+m}(y + Tu_{j_k}) \right\| \leqslant 2^{-n}. \tag{28}$$

Since $y \in T^{-m}(\overline{T^{m+1}(X)})$, we have $T^m y \in \overline{T^{m+1}(X)}$. According to (27), $T^n y \in T^{m+1}(X) + \overline{T^{m+j_1+1}(X)}$. Therefore there exists $w_0 \in X$ for which $T^m y + T^{m+1}w_0 \in \overline{T^{m+j_1+1}(X)}$. Now we can choose $v_0 \in X$ such that $\|T^m y + T^{m+1}w_0 - T^{m+j_1+1}v_0\| \leqslant 1$. Denoting $u_{j_0} = w_0 - T^{j_1}v_0$, we obtain $T^m(y + Tu_{j_0}) \in \overline{T^{m+j_1+1}(X)}$ and $\|T^m(y + Tu_{j_0})\| \leqslant 1$, that is (28) for $n = 0$ is satisfied. The basis of induction is constructed. Suppose now that q is a positive integer and $u_{j_0}, \dots, u_{j_{q-1}}$, satisfying (28) for $n \leqslant q-1$ are already constructed. Denote $x = \sum\limits_{k=0}^{q-1} T^{j_k+m}(y + Tu_{j_k})$. From (28) for $n = q-1$ it follows that $x \in \overline{T^{m+j_q+1}(X)}$. According to (27), $x \in T^{m+j_q+1}(X) + \overline{T^{m+j_{q+1}+1}(X)}$. Since $T^m y \in \overline{T^{m+1}(X)}$, we have $T^{j_{q+1}+m}y \in \overline{T^{m+j_{q+1}+1}(X)}$ and therefore

$$x + T^{j_{q+1}+m}y \in T^{m+j_q+1}(X) + \overline{T^{m+j_{q+1}+1}(X)}.$$

Hence there exists $w_q \in X$ for which

$$x + T^{j_{q+1}+m}y + T^{j_{q+1}+m+1}w_q \in \overline{T^{m+j_{q+1}+1}(X)}.$$

Then we can choose $v_q \in X$ such that

$$\|x + T^{j_{q+1}+m} y + T^{j_{q+1}+m+1} w_q - T^{m+j_{q+1}+1} v_q\| \leqslant 2^{-q}.$$

Denoting $u_{j_q} = w_q - T^{j_{q+1}-j_q} v_q$, we obtain

$$x + T^{j_q+m}(y + Tu_{j_q}) \in \overline{T^{m+j_{q+1}+1}(X)} \quad \text{and} \quad \|x + T^{j_q+m}(y + Tu_{j_q})\| \leqslant 2^{-q},$$

which is exactly (28) for $n = q$. The construction of the sequence $\{u_{j_n}\}_{n\in\mathbb{Z}_+}$, satisfying (28) for each $n \in \mathbb{Z}_+$ is complete. From the inequality in (28) it follows that the partial sums of the series $\sum\limits_{n=0}^{\infty} T^{j_n+m}(y + Tu_{j_n})$ converge to zero with respect to the norm of X. Hence $T^m(y + Tu_0)$ belongs to the closed linear span of

$$\{T^{m+n}(y + Tu_{j_n}) : n \geqslant 1\} = \{T^{m+n}(y + Tu_n) : n \in A \setminus \{0\}\}$$

as required. □

Lemma **7.7.** *Let T be an injective bounded linear operator on a Banach space X such that $\overline{T^{m+1}(X)}$ has finite codimension in $\overline{T^m(X)}$ for some $m \in \mathbb{Z}_+$. Then $B(T, X) = A(T, X)$.*

Proof. The inclusion $B(T, X) \subseteq A(T, X)$ follows from Lemma 7.2. Let $y \in A(T, X) \setminus T(X)$. Since $T^{-n}(\overline{T^{n+1}(X)})$ is an increasing sequence of linear subspaces of X, there exists $q \in \mathbb{Z}_+$ such that $q \geqslant m$ and $y \in T^{-q}(\overline{T^{q+1}(X)})$. According to Lemma 7.4 $\overline{T^{q+1}(X)}$ has finite codimension in $\overline{T^q(X)}$.

Choose a strictly increasing sequence $\{m_k\}_{k\in\mathbb{Z}_+}$ of positive integers and a sequence $\{A_k\}_{k\in\mathbb{Z}_+}$ of infinite subsets of $\mathbb{Z}_+$ such that $m_0 \geqslant q$, $0 \in A_k$ for each $k \in \mathbb{Z}_+$ and the sets $B_k = m_k + A_k = \{m_k + n : n \in A_k\}$ are disjoint. For instance, we can choose a strictly increasing sequence $\{m_n\}_{n\in\mathbb{Z}_+}$ of prime numbers such that $m_0 \geqslant q$ and take $A_k = \{m_k^l - m_k : l = 1, 2, \dots\}$. According to Lemma 7.6, for any $k \in \mathbb{Z}_+$, there exists a sequence $\{u_n\}_{n\in B_k}$ of elements of X such that $T^q(y+Tu_{m_k})$ belongs to the closure of span $\{T^{q+j}(y+Tu_{m_k+j}) : j \in A_k\setminus\{0\}\}$. For $m \in \mathbb{Z}_+ \setminus \bigcup\limits_{k=0}^{\infty} B_k$, we put $u_m = 0$. Since T is continuous, we see that $T^{m_k-l}(y + Tu_{m_k})$ belongs to the closed linear span of $\{T^{m_k-l+j}(y + Tu_{m_k+j}) : j \geqslant 1\}$ if $q + l \leqslant m_k$. From Lemma 7.3 it follows now that the sequence $\{\Phi_n = \Phi_n^{T,\varkappa}\}_{n\in\mathbb{Z}_+}$ of linear functionals on $X_T(\varkappa)$ for $\varkappa = \{y + Tu_n\}_{n\in\mathbb{Z}_+}$ is not weakly bounded. Hence $y \in B(T, X)$. □

Lemma **7.8.** *Let T and S be injective bounded linear operators on Banach spaces X and Y respectively and $G : X \to Y$ be an invertible linear operator such that $SG = GT$. Suppose also that $\overline{T^{m+1}(X)}$ has finite codimension in $\overline{T^m(X)}$ for some $m \in \mathbb{Z}_+$. Then $G(A(T, X)) = A(S, Y)$, $A(T, X)$ is a closed finite codimensional linear subspace of X and $A(S, Y)$ is a closed linear subspace of Y.*

Proof. Since the definition of the set $B(T, X)$ is "algebraic" we have $G(B(T, X)) = B(S, Y)$. By Lemma 7.7, we have $B(T, X) = A(T, X)$. According to Lemma 7.4 $A(T, X)$ is a linear subspace of X of finite codimension, the increasing sequence $T^{-n}(\overline{T^{n+1}(X)})$ of closed linear subspaces stabilizes and $A(T, X) = T^{-n}(\overline{T^{n+1}(X)})$ for sufficiently large n. In particular, $A(T, X)$ is closed in X. Since $G(A(T, X)) = G(B(T, X)) = B(S, Y)$, we see that $B(S, Y)$ is a finite codimensional linear subspace of Y. By Lemma 7.2, $A(S, Y) \supseteq B(S, Y)$ and therefore $A(S, Y)$ has finite codimension in Y. Therefore the increasing sequence $S^{-n}(\overline{S^{n+1}(Y)})$ of closed linear subspaces of Y stabilizes and eventually has finite codimension. From Lemma 7.4 it follows that the codimension of $\overline{S^{n+1}(Y)}$ in $\overline{S^n(Y)}$ is finite for sufficiently large n. Applying Lemma 7.7 once again, we obtain that $B(S, Y) = A(S, Y)$. Since by Lemma 7.4 $A(S, Y) = S^{-n}(\overline{S^{n+1}(Y)})$ for sufficiently large n, we see that $A(S, Y) = B(S, Y)$ is a closed finite codimensional subspace of Y. Finally, putting the above equalities together, we see that $G(A(T, X)) = G(B(T, X)) = B(S, Y) = A(S, Y)$. □

7.1 Proof of Theorem 1.5

Let S be a bounded linear operator on a Banach space Y and $G : X \to Y$ be an invertible linear operator such that $SG = GT$. According to Proposition 1.1 it suffices to prove that G is continuous. Injectivity of T implies injectivity of S.

Consider $H_n \subseteq X$ defined inductively by the formulas

$$H_0 = X, \quad H_{k+1} = A(T, H_k) = \bigcup_{n=0}^{\infty} T^{-n}(\overline{T^{n+1}(H_k)}) \quad \text{for } k \in \mathbb{Z}_+.$$

According to Lemma 7.5 H_k are closed finite codimensional linear subspaces of X and $\bigcap\limits_{k=0}^{\infty} H_k = \{0\}$. Applying Lemma 7.8 consecutively to the restrictions of T to the invariant subspaces H_k, we obtain that $G(H_k)$ are closed in Y. From Lemma 7.1 it now follows that G is continuous. The proof is complete.

8 Concluding remarks

1. Let $\mathcal{H} = \ell_2 \oplus L_2[0,1]$ and $T : \mathcal{H} \to \mathcal{H}$ be the operator defined by the formula

$$T(x \oplus f) = Ax \oplus Vf,$$

where V is the Volterra operator acting on $L_2[0,1]$ and $A : \ell_2 \to \ell_2$ be the weighted forward shift given by $Ae_n = \frac{e_{n+1}}{n+1}$, $\{e_n\}_{n\in\mathbb{Z}_+}$ being the standard orthonormal basis in ℓ_2. Clearly T is bounded injective and quasinilpotent.

Using Proposition 4.4 and Lemma 4.6 it is easy to see that $\dim_{\mathcal{R}} \mathcal{H}_T = \mathfrak{c}$ and $\dim_{\mathbb{C}} \mathcal{H}/T(\mathcal{H}) = \mathfrak{c}$. On the other hand $\overline{T^{n+1}(X)} \neq \overline{T^n(X)}$ for each $n \in \mathbb{Z}_+$. According to Theorem 5.8, T is not tame and therefore not similar to the Volterra operator acting on $C[0,1]$. It also does not determine the topology of $\mathcal{H}$ since it T is the direct sum of the Volterra operator V acting on $L_2[0,1]$ with another operator and V does not determine the topology of $L_2[0,1]$ according to Proposition 1.1 and Theorem 1.2. It worth noting that T satisfies (3) and does not satisfy (2). This leads to the following natural question.

Problem 1. *Let T be a bounded injective linear operator on a Banach space X such that* (2) *is satisfied. Is it true that T determines the topology of X?*

2. A characterization of similarity of linear operators on finite dimensional vector spaces is provided by the Jordan block decomposition theorem, from which it follows that the spectrum $\sigma(T)$ and the dimensions of $\ker(T - \lambda I)^n$ for $\lambda \in \sigma(T)$, $n \in \mathbb{Z}_+$ are similarity invariants, which determine T up to similarity. This is obviously not true in infinite dimensional case, when we have that the codimensions of $(T - \lambda I)^n(X)$ and of $\bigcap\limits_{k=0}^{\infty} (T - \lambda I)^k(X)$, which are also similarity invariants, are not determined by the first family of invariants. The natural conjecture that these dimensions and codimensions altogether form a set of invariants determining T up to similarity fails dismally as follows from Theorem 2.2, as well as from the above example. This leads us to the following problem.

Problem 2. *Characterize similarity of linear operators on infinite dimensional vector spaces.*

3. It worth noting that under the continuum hypothesis Corollary 4.3, Proposition 4.4, Lemma 4.5 and Lemma 4.6 become trivial consequences of the Baire Theorem. However if one does not assume

the continuum hypothesis the Baire category argument fails, since it is compatible with ZFC[3] that a separable infinite dimensional Banach space is a union of less than continuum of compact subsets.
4. The class of operators satisfying the conditions (2) and (3) of Theorem 1.5 is rather rich. For instance, it contains bounded weighted forward shifts on the spaces ℓ_p, $1 \leqslant p \leqslant \infty$ and the Volterra operator acting on the Hardy spaces $\mathcal{H}^p$ of the unit disk for $1 \leqslant p \leqslant \infty$. In particular, it contains plenty of injective quasinilpotent operators. This class is also closed under finite powers and finite direct sums. It worth noting that (2) and (3) imply that there are countably many continuous linear functionals on X separating points and therefore the algebraic dimension of X is $\mathfrak{c}$.
5. The following theorem is proved in [7].

THEOREM **J.** *Let A be a commutative unital semisimple Banach algebra and $a \in A$. Then the multiplication operator $M_a : A \to A$, $M_a x = ax$ determines the topology of A if and only if for any $\lambda \in \mathbb{C}$ either $a - \lambda$ is not a zero divisor or $a - \lambda$ is a zero divisor and the codimension of $(M_a - \lambda I)(A)$ in A is finite.*

The classes of operators on Banach spaces determining the topology, provided by Theorem J and by Theorem 1.5 do not cover each other (although they do intersect). For instance, the multiplication operator $T : C[0,1] \to C[0,1]$, $Tf(x) = xf(x)$ determines the topology of $C[0,1]$ according to Theorem J and it does not satisfy (2). On the other hand a multiplication operator M_a with $a \neq 0$ on a commutative unital semisimple Banach algebra is never quasinilpotent, while Theorem 1.5 can be applied to certain quasinilpotent operators like the Volterra operator acting on the Hardy space $\mathcal{H}^p$ of the unit disk. It would be interesting to find a unified approach, generalizing Theorems J and 1.5 simultaneously.

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